Journal of Applied Science and Education, 2025,
Vol. 05, Iss. 02, S. No. 112, pp. 1-9
ISSN (Online): 2583-1372

Stability and Perturbations of Chi-Phi Frame in Banach Spaces

Ram Bharat Singh[1], Sandeep Kumar[2], Anil Kumar[3]

[1]Department of Applied Sciences & Humanities (Mathematics), IMS Engineering College, Ghaziabad (U.P.) - 201015, India
[1]rambharat.maths@gmail.com, [2]sandeep.kalher13@gmail.com, [3]dranilkumar73@rediffgmail.com

How to cite this paper: R. B. Singh, S. Kumar and A. Kumar, "Stability and Perturbations of χ_ϕ-Frame in Banach Spaces," *Journal of Applied Science and Education (JASE)*, Vol. 05, Iss. 02, S. No. 0112, pp 1-9, 2025.

https://doi.org/10.54060/a2zjournals.jase.112

Received: 15/02/2025
Accepted: 17/06/2025
Online First: 14/07/2025
Published: 14/07/2025

Abstract

Frames in relation to certain sequence spaces, specifically Chi-Phi frames, were introduced and studied. Further investigations into Chi-Phi frames and their various characterizations can be found. The stability and perturbation of frames are crucial in practical applications and have been extensively studied. In this paper, we examine the stability and perturbations of Chi-Phi frames and Chi-Phi Bessel sequences in Banach spaces, extending two important propositions from [3]and [7], respectively.

2000 AMS Subject Classification: *42C15, 42A38*

Keywords

Stability, Analysis operator, Perturbations, χ_ϕ-Bessel sequence and χ_ϕ-frames

1. Introduction

For more than a century, a key tool for analysis has been the Fourier transform. Its inability to demonstrate phase information relevant to a signal's width and point of emission is an issue in signal analysis. In order to solve this, a localized time-frequency representation that encodes this crucial information was developed.

Dennis Gabor [10] covered this gap in 1946 by establishing an elementary approach to signal decomposition using elementary signals. Building on this progress, Duffin and Schaeffer [8] stated the concept of a frame for Hilbert spaces in 1952. This frame is defined as follows:

A system of non-zero vectors $\{y_n\}_{n=1}^{\infty} \subset \mathrm{H}$ be known a frame in H (separable Hilbert space) if $\exists$ non-zero the constants $0 < a \le b < \infty$ satisfying the following

$$a\|y\|_{\mathrm{H}}^2 \le \sum_{n=1}^{\infty} |\langle y, y_n\rangle|^2 \le b\|y\|_{\mathrm{H}}^2, \quad \forall\ y \in \mathrm{H} \tag{1.1}$$

where $\|\bullet\|$ denotes norm and $\langle \bullet, \bullet \rangle$ represent inner product in H. These above positive constants a & b in frame inequality (1.1) are known as the lower and upper frame bound for $\{x_n\}_{n=1}^{\infty}$ respectively. The positive ratio $\kappa = a/b$ is called the condition coefficient of the frame. If $\kappa = 1$, the frame is referred to as a tight frame. Developments in frame theory for Hilbert spaces have led to the extension of known results to the Banach space setting. The generalization of frames to Banach spaces has been studied extensively [1, 9, 11, 12, 15, 18, 19]. Frames share many properties with bases but lack a crucial one uniqueness which makes them highly useful in various applications, including function spaces, signal and image processing, filter banks, wireless communications, and more.

The $\mathcal{X}_d$ -frames, was introduced in [1] and generalizes p -frames which were widely studied in [4]. In the present manuscripts, we examine the stability and perturbations of χ_ϕ-Bessel sequences and χ_ϕ-frames in Banach spaces by generalizing two fundamental results, presented in the form of propositions from [7] and [3], respectively.

2. Preliminaries

In the present paper, we generally use the character χ to describe an infinite-dimensional Banach space over the scalar field K (R or C). Let χ^* the first conjugate spaces of χ.. Additionally, let ℒ (χ , F) be the Banach space of all bounded linear transformations from χ into F.

We introduce the following definitions, which will be frequently used throughout this investigation.

Definition 2.1 ([1]). If $_d$ be a sequence space which is a Banach space and its coordinate functionals are continuous, then it is commonly referred to as a BK-space. i.e., $y_m = \{\beta_k^{(m)}\}$, $y = \{\beta_k\} \in \mathcal{X}_d$, $\lim_{m\to\infty} y_m = y \Rightarrow \underset{m\to\infty}{Lt}\, \beta_k^{(m)} = \beta_k\ (k = 1, 2, \ldots\ldots).$

The theory of scalar sequence spaces naturally extends to vector sequence spaces. If **ϕ**= $\{Z_n\}$ be collection of a sequence of Banach spaces, then χ_ϕ associated with $\{Z_n\}$ will be a linear subspace of $\prod_{n=1}^{\infty} Z_n$.

The coordinate transformations $\mathrm{T_m} : \mathbf{X}_\Phi \to Z_n$ are defined by

$$\mathrm{T_m}\left(\{y_j\}\right) = y_m,\ m = 1, 2, \ldots$$

A space χ_ϕ is termed as commonly BK-space prompted by $\{Z_n\}$ then χ_ϕ becomes a Banach space and each $\mathrm{T_m}$ will be bounded transformation. The BK-spaces which involves all normal vectors $\{e_n\}$ generalized by χ_ϕ spaces that involves subspaces given by

$$H_m = \{0\} \times \{0\} \times \ldots\ldots\ldots\ldots \{0\} \times \underset{m^{th}\ place}{\underset{\downarrow}{Z_m}} \times \{0\} \times \ldots\ldots\ldots\ldots \left(Z_m \ne \{0\},\ m = 1, 2\ldots\ldots\right).$$

These H_m's form closed linear spaces which are subspaces of χ_ϕ, and the termed χ_ϕ as a mode sequence space.

Example 2.2 We discuss an example of a Model Space.

Suppose $\boldsymbol{\phi}=\{Z_n\}$ be collection of a sequences of closed linear subspaces of a Banach space E. Let us assume that χ_ϕ of collection $\boldsymbol{\phi}$, i.e., defined as the space of all sequences $\{z_n\}_{n=1}^{\infty}$ such that each $z_n \in Z_n$ and $\sum_{n=1}^{\infty} z_n$ be convergent in χ equipped with the norm

$$\left\|\{z_n\}_{n=1}^{\infty}\right\|_{\chi_\phi} = \operatorname*{Sup}_{1\le n<\infty}\left\{\left\|\sum_{i=1}^{n} z_i\right\|_{\chi} : z_n \in Z_n\,(n=1,2...)\right\}. \tag{2.1}$$

It follows that χ_ϕ will be complete with reference to the norm (2.1). To verify this, we first observe that (2.1) defines a valid norm on χ_ϕ. Now, let $\left\{z_n^{(k)}\right\}_{k=1}^{\infty}$ be a Cauchy sequence in χ_ϕ. Then, for every $\varepsilon>0$, $\exists$ $\mathbf{N}(\varepsilon)$ such that, for all $k,m>\mathbf{N}(\varepsilon)$,

$$\left\|\left\{z_n^{(k)}\right\}-\left\{z_n^{(m)}\right\}\right\|_{\chi_\phi} < \varepsilon. \tag{2.2}$$

This indicates that for each n the sequence $\left\{z_n^{(k)}\right\}_{k=1}^{\infty}$ becomes a Cauchy sequence in Z_n. But each Z_n are complete, so there exists $z_n \in Z_n$ such that $\left\{z_n^{(k)}\right\}_{k=1}^{\infty} \to \{z_n\}$ as $k\to\infty$.
From the inequality (2.2), we deduce that for all $k,m>\mathbf{N}(\varepsilon)$,

$$\operatorname*{Sup}_{1\le n<\infty}\left\|\sum_{i=1}^{n}\left(z_i^{(k)}-z_i^{(m)}\right)\right\|<\varepsilon$$

Thus, making the limit as $m\to\infty$, we obtain

$$\operatorname*{Sup}_{1\le n<\infty}\left\|\sum_{i=1}^{n}\left(z_i^{(k)}-z_i\right)\right\|<\varepsilon \qquad (k>\mathbf{N}(\varepsilon), n=1,2.....)$$

This show that the series $\sum_{i=1}^{\infty} z_i^{(k)}$ converges in χ but χ is complete so we have $\{z_i\}\in\chi_\phi$. Moreover, we have $\left\|\left\{z_n^{(k)}\right\}-\{z_n\}\right\|_{X_\Phi} \le \varepsilon \quad \left(k>\mathbf{N}(\varepsilon)\right)$, which show that $\left\{z_n^{(k)}\right\}\to\{z_n\}$ in χ_ϕ. Thus, χ_ϕ is complete with respect to the norm (2.1).

The system $\{H_m\}$, as defined above, forms a Schauder decomposition of χ_ϕ. Clearly, any model space χ_ϕ can be obtained using this method. Indeed, if χ_ϕ is a model space of a sequence of subspaces $\boldsymbol{\phi}=\{Z_n\}$, then χ_ϕ can be constructed as described above.

Definition 2.3. Suppose χ is a Banach space over F & χ_ϕ is a model space induced by $\{Z_n\}$. For all $n\in\mathbf{N}$, $\{w_n\}$ is a sequence of bounded linear operators in $\mathcal{L}(\chi, Z_n)$. Then the collection $\mathrm{W}=\{w_n\}$ is said to be χ_ϕ-Bessel sequence for χ corresponding to χ_ϕ, $\exists$ a positive constant $b>0$ exists such that

$$\left\|\{w_n(y)\}\right\|_{\chi_\phi} \le b\|y\|_\chi, \quad \forall x\in\chi.$$

The constant b is called Bessel bound for T.

Define, $b_w = Inf\left\{b>0:\left\|\{w_n(y)\}\right\|_{\chi_\phi} \le b\|y\|_E, \forall\, x\in\chi\right\}$.

Then this constant b_w is called optimum Bessel bound of W.

For any $y\in\chi$, define an operator $R_W:\chi\to\chi_\phi$ such that $R_W(y)=\{w_n(y)\}$.

This operator R_W is called the analysis operator for W. Clearly, from this definition, we obtain $R_W\in \mathcal{L}(\chi,\chi_\phi)$.

Definition 2.4. Suppose $\boldsymbol{\phi} = \{Z_n\}$ be a collection of non-trivial subspaces of Banach space E. Let $T=\{w_n : w_n\in \mathbf{L}(\chi,Z_n), \forall\, n\in\mathbf{N}\}$ is a collection of sequence of linear operators from χ to Z_n and let χ_ϕ is a model space corresponding to χ. Thus the pair $(\{Z_n\},\{w_n\})$ is known as χ_ϕ-frame for χ corresponding to χ_ϕ satisfying the following properties:

(i) $\{w_n(y)\}\in\chi_\phi, \quad \forall y\in\chi$

(ii) $\exists$ two constants a, b>0 with $a\le b<\infty$ such that

$$a\|y\|_\chi \le \left\|\{w_n(y)\}\right\|_{\chi_\phi} \le b\|y\|_\chi,\ \forall\ y\in E.$$

These above numbers b and a are known as upper and lower bounds, respectively for χ_ϕ-frame, respectively.

Define, $a_T = Sup\left\{a>0: a\|y\|_\chi \le \left\|\{w_n(y)\}\right\|_{\chi_\phi}, \forall\, y\in\chi\right\}$

and $b_T = Inf\left\{b>0:\left\|\{w_n(y)\}\right\|_{\chi_\phi} \le b\|y\|_\chi, \forall\, y\in\chi\right\}$.

These constants a_T and b_T are known as lower and upper optimum bounds of $T=\{w_n\}$, respectively.

Definition 2.5. The χ_ϕ-frame for χ corresponding to χ_ϕ, is known as a tight frame if $a_T=b_T$. Moreover, if $a_T=b_T=1$, then we call that the family $T=\{w_n\}$ is a Parseval (or Normalized tight) χ_ϕ-frame for E corresponding to χ_ϕ.

3. Main Results

In this section, we study stability and perturbations of χ_ϕ-frames. The perturbations of frames are of great practical importance and have been widely studied in the various literature [6, 7, 21, and 22]. Below are two fundamental results concerning to frame perturbations.

Proposition 3.1. ([7]). Suppose H is a Hilbert Space and $\{u_n\}$ is a frame for H, with respectively, bounds a, b. Then the family $\{v_n\}$ of vectors in H satisfying following property

$$\sum_{n=1}^{\infty}\|u_n-v_n\|^2 < a$$

forms a frame for H with bounds $a\left[1-\sqrt{\dfrac{\sum_{n=1}^{\infty}\|u_n-v_n\|^2}{a}}\right]^2$ and $b\left[1-\sqrt{\dfrac{\sum_{n=1}^{\infty}\|u_n-v_n\|^2}{b}}\right]^2$.

Proposition 3.2. ([3]). Suppose $\{u_i\}$ is a frame for Hilbert space H, with bounds a, b. If $\{v_i\}\subset\mathrm{H}$ and $\exists$ non-negative numbers α,β and $\theta\ge 0$, which satisfying following two conditions

(i) $\max\left\{\alpha+\frac{\theta}{\sqrt{a}},\beta\right\}<1$ and

(ii) $\left\|\sum_{i=1}^{n}d_i\left(u_i-v_i\right)\right\| \leq \alpha\left\|\sum_{i=1}^{n}d_iu_i\right\| + \beta\left\|\sum_{i=1}^{n}d_iv_i\right\| + \theta\left(\sum_{i=1}^{n}\left|d_i\right|^2\right)^{\frac{1}{2}}$

for all $d_1,d_2,...,d_n\left(n\geq 1\right)$. Then $\left\{v_i\right\}$ forms a frame for H with frame bounds

$$a\left[1-\frac{\alpha+\beta+\frac{\theta}{\sqrt{a}}}{1+\alpha}\right]^2 \text{ and } b\left[1-\frac{\alpha+\beta+\frac{\theta}{\sqrt{a}}}{1+\alpha}\right]^2, \text{ respectively.}$$

Theorems given below examine the stability and perturbations of χ_ϕ-Bessel sequences as well as χ_ϕ-frames in Banach spaces, extending the previously stated propositions to the context of χ_ϕ-Bessel sequences as well as χ_ϕ-frames in Banach spaces.

Theorem 5.3. Suppose $T=\left\{w_n\right\}$ is a χ_ϕ-frame for E corresponding to χ_ϕ with frame bounds a_T, b_T. If $S=\left\{s_n\right\}$ forms a χ_ϕ-Bessel sequence for χ corresponding to χ_ϕ with bounds m such that $m<a_T$. Then, $\left\{w_n\pm s_n\right\}$ forms a χ_ϕ-frame for χ corresponding to χ_ϕ.

Proof. Given that $T=\left\{w_n\right\}$ is a χ_ϕ-frame for E with frame bounds a_T, b_T. We have

$$a\|y\|_\chi \leq \left\|\left\{w_n\left(y\right)\right\}\right\|_{\chi_\phi} \leq b\|y\|_\chi,\ \forall\, y\in\chi.$$

Similarly, since it is also given $S=\left\{s_n\right\}$ is a χ_ϕ-Bessel sequence for χ with bounds $m<a_T$, we have

$$\left\|\left\{s_n\left(y\right)\right\}\right\|_{\chi_\phi} \leq m\|y\|_\chi,\ \forall\, y\in\chi.$$

If R_T, R_S are the analysis operators for $\left\{w_n\right\}$ and $\left\{s_n\right\}$, respectively, then $y\in\chi$, we get $R_T\left(y\right)=\left\{w_n\left(y\right)\right\}$,

and $R_S\left(y\right)=\left\{s_n\left(y\right)\right\}$.

For any $y\in\chi$,

$$\begin{aligned}\left\|\left\{\left(w_n\pm s_n\right)\left(y\right)\right\}\right\|_{\chi_\phi} &= \left\|\left\{w_n\left(y\right)\pm s_n\left(y\right)\right\}\right\|_{\chi_\phi}\\ &= \left\|R_T\left(y\right)\pm R_S\left(y\right)\right\|_{\chi_\phi}\\ &\leq \left(b_T+m\right)\|y\|_\chi,\forall y\in\chi.\end{aligned}$$

Thus, $\left\{w_n\pm s_n\right\}$ forms a χ_ϕ-Bessel sequence for E with bound $\left(b_T+m\right)$.

By further analysis, we establish

$$\begin{aligned}\left\|R_{T\pm S}\left(y\right)\right\|_{\chi_\phi} &= \left\|\left\{\left(w_n\pm s_n\right)\left(x\right)\right\}\right\|_{\chi_\phi}\\ &= \left\|R_T\left(y\right)\pm R_S\left(y\right)\right\|_{\chi_\phi}\\ &\geq \left(a_T-m\right)\|y\|_\chi,\forall y\in\chi.\end{aligned}$$

On behalf of above both inequalities, we must have

$$\left(a_T - m\right)\|y\|_{\chi} \leq \left\|\left\{\left(w_n \pm s_n\right)(y)\right\}\right\|_{\chi_\phi} \leq \left(b_T + m\right)\|y\|_{\chi}, \ \forall y \in \chi.$$

Hence, $\{w_n \pm s_n\}$ forms a χ_ϕ-frame for χ with, respectively frame bounds $(a_T - m)$ and $(b_T + m)$.

Theorem 5.4. Suppose $T = \{w_n\}$ is a χ_ϕ-frame for E corresponding to model space χ_ϕ with frame bounds a_T, b_T. For each $n \in \mathbf{N}$, $s_n \in \mathbf{L}(\chi, Z_n)$ and α, β and $\theta \geq 0$ which satisfying following two conditions

(i) max $\left\{\alpha + \dfrac{\theta}{\sqrt{a}}, \beta\right\} < 1$ and (ii) $\left\|\left\{\left(w_n - s_n\right)(y)\right\}\right\|_{\chi_\phi} \leq \alpha\left\|\left\{w_n(y)\right\}\right\|_{\chi_\phi} + \beta\left\|\left\{s_n(y)\right\}\right\|_{\chi_\phi} + \theta\|y\|_{\chi}, \forall y \in \chi.$

(3.1)

Then, above sequence $\{s_n\}$ forms a χ_ϕ-frame for χ with respectively, frame bounds $\left[\dfrac{(1-\alpha)a_T - \theta}{1+\beta}\right]$ and $\left[\dfrac{(1+\alpha)b_T + \theta}{1-\beta}\right]$.

Proof. It is given $T = \{w_n\}$ be a χ_ϕ-frame for χ with respectively frame bounds a_T, b_T. So we obtain

$$a_T\|y\|_{\chi} \leq \left\|\left\{w_n(y)\right\}\right\|_{\chi_\phi} \leq b_T\|y\|_{\chi}, \ \forall y \in \chi.$$

Suppose F be the finite subset of set of natural numbers $\mathbf{N}$. Define

$$w_n' = \begin{cases} w_n, & \text{when } n \in \mathrm{F} \\ 0, & \text{when } n \in \mathbf{N} - \mathrm{F} \end{cases}$$

and

$$s_n' = \begin{cases} s_n, & \text{when } n \in \mathrm{F} \\ 0, & \text{when } n \in \mathbf{N} - \mathrm{F} \end{cases}$$

From this definition we have, $\{s_n'(y)\} \in \chi_\phi$ and

$$\begin{aligned} \left\|\left\{\left(w_n - s_n\right)(y)\right\}\right\|_{\chi_\phi} &= \left\|\left\{\left(w_n' - s_n'\right)(y)\right\}\right\|_{\chi_\phi} \\ &= \left\|\left\{s_n'(y)\right\} - \left\{w_n'(y)\right\}\right\|_{\chi_\phi} \\ &\geq \left\|\left\{s_n'(y)\right\}\right\|_{\chi_\phi} - \left\|\left\{w_n'(y)\right\}\right\|_{\chi_\phi} \end{aligned}$$

Thus, we get

$$\left\|\left\{\left(w_n - s_n\right)(y)\right\}\right\|_{\chi_\phi} \geq \left\|\left\{s_n'(y)\right\}\right\|_{\chi_\phi} - \left\|\left\{w_n'(y)\right\}\right\|_{\chi_\phi} \qquad (3.2)$$

From inequalities (3.1) and (3.2), we have

$$\begin{aligned} \left\|\left\{s_n'(y)\right\}\right\|_{\chi_\phi} - \left\|\left\{w_n'(y)\right\}\right\|_{\chi_\phi} &\leq \alpha\left\|\left\{w_n(y)\right\}\right\|_{\chi_\phi} + \beta\left\|\left\{s_n(y)\right\}\right\|_{\chi_\phi} + \theta\|y\|_{\chi} \\ &\leq \alpha\left\|\left\{w_n'(y)\right\}\right\|_{\chi_\phi} + \beta\left\|\left\{s_n'(y)\right\}\right\|_{\chi_\phi} + \theta\|y\|_{\chi} \end{aligned}$$

Thus, we have

$$(1-\beta)\left\|\left\{s_n'(y)\right\}\right\|_{\chi_\phi} \leq (1+\alpha)\left\|\left\{w_n'(y)\right\}\right\|_{\chi_\phi} + \theta\|y\|_{\chi}$$

$$\left\|\{s'_n(y)\}\right\|_{\chi_\phi} \le \left(\frac{1+\alpha}{1-\beta}\right)\left\|\{w'_n(y)\}\right\|_{\chi_\phi} + \frac{\theta}{1-\beta}\|y\|_\chi$$
$$\le \left(\frac{1+\alpha}{1-\beta}\right)\left\|\{w_n(y)\}\right\|_{\chi_\phi} + \frac{\theta}{1-\beta}\|y\|_\chi$$
$$\le \left(\frac{1+\alpha}{1-\beta}\right) b_T\|y\|_\chi + \frac{\theta}{1-\beta}\|y\|_\chi$$
$$= \left[\frac{(1+\alpha)b_T+\theta}{1-\beta}\right]\|y\|_\chi .$$

Consequently, we get

$$\left\|\{s_n(y)\}\right\|_{\chi_\phi} = \left\|\{s'_n(y)\}\right\|_{\chi_\phi} \le \left[\frac{(1+\alpha)b_T+\theta}{1-\beta}\right]\|y\|_\chi$$

Let $\mathrm{F}\to\mathbf{N}$ and for each $y\in\chi$, we get $\left\|\{v_n(y)\}\right\|_{\chi_\phi} \le \left[\frac{(1+\alpha)b_T+\theta}{1-\beta}\right]\|y\|_\chi .$ (3.3)

This show that $\{s_n\}$ forms a χ_ϕ-Bessel sequence for with bounds $\left[\frac{(1+\alpha)b_T+\theta}{1-\beta}\right]$.

Furthermore as similar above, we get

$$\left\|\{(w_n-s_n)(y)\}\right\|_{\chi_\phi} = \left\|\{(w'_n-s'_n)(y)\}\right\|_{\chi_\phi}$$
$$= \left\|\{w'_n(y)\}-\{s'_n(y)\}\right\|_{\chi_\phi}$$
$$\ge \left\|\{w'_n(y)\}\right\|_{\chi_\phi} - \left\|\{s'_n(y)\}\right\|_{\chi_\phi}$$

Thus, we obtain $\left\|\{(w_n-s_n)(y)\}\right\|_{\chi_\phi} \ge \left\|\{w'_n(y)\}\right\|_{\chi_\phi} - \left\|\{s'_n(y)\}\right\|_{\chi_\phi}$ (3.4)

Clearly from inequalities (3.4) and (3.1), we get

$$\left\|\{w'_n(y)\}\right\|_{\chi_\phi} - \left\|\{s'_n(y)\}\right\|_{\chi_\phi} \le \alpha\left\|\{w_n(y)\}\right\|_{\chi_\phi} + \beta\left\|\{s_n(y)\}\right\|_{\chi_\phi} + \theta\|y\|_\chi$$
$$\le \alpha\left\|\{w'_n(y)\}\right\|_{\chi_\phi} + \beta\left\|\{s'_n(y)\}\right\|_{\chi_\phi} + \theta\|y\|_\chi$$

Thus, we have

$$(1+\beta)\left\|\{s_n'(y)\}\right\|_{\chi_\phi} \geq (1-\alpha)\left\|\{w_n'(y)\}\right\|_{\chi_\phi} - \theta\|y\|_\chi$$

$$\begin{aligned}\left\|\{s_n'(y)\}\right\|_{\chi_\phi} &\geq \left(\frac{1-\alpha}{1+\beta}\right)\left\|\{w_n'(y)\}\right\|_{\chi_\phi} - \frac{\theta}{1+\beta}\|y\|_\chi \\ &\geq \left(\frac{1-\alpha}{1+\beta}\right)\left\|\{w_n(y)\}\right\|_{\chi_\phi} - \frac{\theta}{1+\beta}\|y\|_\chi \\ &\geq \left(\frac{1-\alpha}{1+\beta}\right)a_T\|y\|_\chi - \frac{\theta}{1+\beta}\|y\|_\chi \\ &= \left[\frac{(1-\alpha)a_T-\theta}{1+\beta}\right]\|y\|_\chi\end{aligned}$$

Consequently, we get

$$\left\|\{s_n(y)\}\right\|_{\chi_\phi} = \left\|\{s_n'(y)\}\right\|_{\chi_\phi} \geq \left[\frac{(1-\alpha)a_T-\theta}{1+\beta}\right]\|y\|_\chi$$

Let $\mathrm{F} \to \mathbf{N}$ and for each $y \in \chi$, we get $\left\|\{s_n(y)\}\right\|_{\chi_\phi} \geq \left[\frac{(1-\alpha)a_T-\theta}{1+\beta}\right]\|y\|_\chi$. (3.5)

Therefore from inequalities (3.3) and (3.5), we have

$$\left[\frac{(1-\alpha)a_T-\theta}{1+\beta}\right]\|y\|_\chi \leq \left\|\{s_n(y)\}\right\|_{\chi_\phi} \leq \left[\frac{(1+\alpha)b_T+\theta}{1-\beta}\right]\|y\|_\chi, \forall\, y \in \chi.$$

Hence, $\{s_n\}$ forms a χ_ϕ-frame for χ with, respectively frame bounds $\left[\frac{(1-\alpha)a_T-\theta}{1+\beta}\right]$ and $\left[\frac{(1+\alpha)b_T+\theta}{1-\beta}\right]$.

4. Conclusion

In this paper, we investigated the stability and perturbations of χ_ϕ-frames and χ_ϕ- Bessel sequence in Banach spaces, generalizing of the propositions [3] and [7]. Furthermore, we provided a Paley-Wiener type perturbation theorem for χ_ϕ-Banach frames in Banach space setting. These results extend the classical frame perturbation theory to a more general framework.

References

[1]. P. G. Casazza, O. Cristenson, and D. T. Stoeva, "Frames expansions in separable Banach spaces," *J. Math.,Anal. Appl*, vol. 307, no. 2, pp. 710–723, 2005.

[2]. P. G. Casazza, D. Han, and D. R. Larson, "Frames for Banach spaces, Contemp," *Contemp. Math*, vol. 247, pp. 149–182, 1999.

[3]. P. G. Casazza and O. Christensen, "Perturbation of operators and applications to frame theory," *J. Fourier Anal. Appl*, vol. 3, no. 5, pp. 543–557, 1997.

[4]. O. Christensen and D. T. Stoeva, "-Frames in separable Banach spaces, Adv," *Adv. Comput.Math*, vol. 18, no. 2, pp. 117–126, 2003.

[5]. O. Christensen and C. Heil, “Perturbations of Banach frames and atomic decompositions: Perturbations of frames,” *Math. Nachr.*, vol. 185, no. 1, pp. 33–47, 1997.

[6]. O. Christensen, “An itroduction to frames and Riesz bases,” *Appl. Numer. Harmon. Anal*, 1992.

[7]. O. Christensen, “Frame perturbation,” *Proc. Amer. Math. Soc*., vol. 123, no. 4, pp. 1217-1220, 1995.

[8]. R. J. Duffin and A. C. Schaeffer, “A class of non-harmonic Fourier series,” *Trans. Amer. Math. Soc*, vol. 72, pp. 341–366, 1952.

[9]. H. G. Feichinger and K. Gröchenig, “A unified approach to atomic decompositions via integrable group representations,” in *Proc. Conf. ``Function Spaces and Applications*, vol. 1302, Berlin - Heidelberg - New York: Springer, pp. 52–73, 1988.

[10]. D. Gabor, “Theory of communications,” *J.I.E.E(London)*, vol. 93, no. 3, pp. 429–457, 1946.

[11]. K. Gröchenig, “Describing functions., Atomic decompositions versus frames,” *Monatsh. Math*, vol. 112, pp. 1–41, 1991.

[12]. D. Han and D. R. Larson, “Frames, bases and group representation for Banach space,” *Memoirs, Amer. Math. Soc*, vol. 147, pp. 1–91, 2000.

[13]. P. K. Jain, S. K. Kaushik, and L. K. Vashisht, “On stability of Banach frames, Bull,” *Bull. Korean Math*, vol. 37, no. 112, pp. 1–41, 2001.

[14]. P. K. Jain, S. K. Kaushik, and L. K. Vashisht, “On stability of Banach frames, Bull,” *Bull. Korean Math*, vol. 37, no. 112, pp. 1–41, 2001.

[15]. M. Joshi, R. Kumar, and R. Singh, “On a weighted retro Banach frames for discrete signal spaces,” *Br. J. Math. Comput. Sci.*, vol. 4, no. 23, pp. 3334–3344, 2014.

[16]. R. Kumar, M. C. Joshi, R. B. Singh, and A. K. Sah, “Construction of generalized atomic decomposi-tions in Banach spaces,” *Int. J. Adv. Math. Sci*, vol. 2, no. 3, pp. 116–124, 2014.

[17]. A. K. Sabherwal, R. B. Singh, and R. Kumar, “A note on -frame in Banach spaces,” *Advances in Theoretical and Applied Mathematics*, vol. 11, no. 4, pp. 495–504, 1016.

[18]. I. Singer, “Bases in Banach spaces-II,” *Springer, New York,* 1970.

[19]. R. B. Singh, M. Joshi, and A. K. Sah, “A note on -frames in Banach spaces,” *Int. J. Math. Arch*, vol. 5, no. 3, pp. 1–6, 2014.

[20]. R. B. Singh, *On Generalizations of frames in Banach spaces*. Nainital, India, 2014.

[21]. W. Sun, Stability of g-frames,” *J. Maths Anal. Appl*., vol. 326, pp. 858-868, 2007..

[22]. R. M. Young, *An Introduction to Non-harmonic Fourier series*. New York: Academic Press, 1980.