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Solving a Birkhoff Interpolation Problem for (0, 1, 5) Data

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Abstract

The following research work deals with a special type of Birkhoff's interpolation problem in which we have 3 sets of data prescribed in the unit interval I = [0, 1]. Data values are the function value, first derivatives and fifth derivatives prescribed at nodes of the unit interval. We obtained a unique spline interpolating the given data along with the convergence problem.

Keywords

Spline function, Lacunary Interpolation, Birkhoff interpolation, AMS Classification: 65L10

1. Introduction

In this study, we use virtually quintic splines $s(x) \in S^{(2)}_{n,5}(x)$ for a given partition Δ to solve a Birkhoff interpolation issue we term (0, 1, 5) problem. Let Δ : 0 = x_0, x_1.......... x_{n-1}, x_n=1 be a partition of the unit interval I = [0,1] and $S^{(2)}_{n,5}(x)$ denotes the class of spline function s(x) such that

(1.1) $s_i(x) \in \pi_5$, i = 0(1)n-2

$s_i(x) \in \pi_6$, i= 0(1)n −1 where x∈ $[x_i, x_{i+1}]$

(1.2) s(x) ∈ C^2 (I).

Also, we denote $x_{i+1} - x_i = h_i$, for all i = 0(1)n −1. We prove the existence and uniqueness of such spline functions and show that they converge to the given function $f(x) \in C^5$ (I) up to derivative of order 5. For relevant reading one is referred to [1 -7].

Following two theorems were proved as given in the section 2 and 3.

2. Theorem of Unique Existence

Given Δ and the real numbers $y_i,\ y_i',\ y_i^{(5)},\ i = 0(1)n,\ y_0'',\ y_n''$ there exists unique $s_\Delta(x) \in S^{(2)}_{n,5}(x)$ such that

(2.1) $s_\Delta^{(q)}(x_i) = y_i^{(q)}, \quad i = 0(1)n-2 \quad q = 0,1,5$

(2.2) $s_\Delta''(x_0) = y_0'', \qquad s_\Delta''(x_n) = y_n''$

Proof:

We set

(2.3) $$s_\Delta(x) = \begin{cases} s_0(x), where\ x \in [x_0, x_i] \\ s_i(x), when\ x \in [x_i, x_{i+1}] \ \text{i=01(1)n-2.} \\ s_{n-1}(x), when\ x \in [x_{n-1}, x_n] \end{cases}$$

Usinginterpolatory conditions we write,

(2.4) $$s_0(x) = y_0 + \frac{(x-x_0)}{1!}y_0 + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}a_{0,3} + \frac{(x-x_0)^4}{4!}a_{0,4} + \frac{(x-x_0)^5}{5!}y_0^{(5)} + \frac{(x-x_0)^6}{6!}a_{0,6}$$

(2.5) $$s_i(x) = y_i + \frac{(x-x_i)}{1!}y_i^1 + \frac{(x-x_i)^2}{2!}a_{i,2} + \frac{(x-x_i)^3}{3!}a_{i,3} + \frac{(x-x_i)^4}{4!}a_{i,4} + \frac{(x-x_i)^5}{5!}y_i^{(5)}$$

(2.6)

$$s_{n-1}(x) = y_{n-1} + \frac{(x-x_{n-1})}{1!}y_{in-1} + \frac{(x-x_{n-1})^2}{2!}a_{n-1,2} + \frac{(x-x_{n-1})^3}{3!}a_{n-1,3} + \frac{(x-x_{n-1})^4}{4!}a_{n-1,4} + \frac{(x-x_{n-1})^5}{5!}y_{n-1}^{(5)} + \frac{(x-x_{x-1})^6}{6!}a_{n-1,6}$$

The coefficients involved in the above equation are determined by the remaining interpolatory conditions and the continuity requirement that $s_\Delta(n) \in C^2(I)$. Applying this condition we get the following set of equations.

(2.7) $$\begin{cases} y_1 = y_0 + h_0 y_0' + \frac{h_0^2}{2!}y_0'' + \frac{h_0^3}{3!}a_{0,3} + \frac{h_0^4}{4!}a_{0,4} + \frac{h_0^5}{5!}y_0^5 + \frac{h_0^6}{6!}a_{0,6} \\ y_1' = y_0' + h_0 y_0'' + \frac{h_0^2}{2!}a_{0,3} + \frac{h_0^3}{3!}a_{0,4} + \frac{h_0^4}{4!}y_0^{(5)} + \frac{h_0^5}{5!}a_{0,6}, \\ y_1^5 = y_0^{(5)} + h_0 a_{0,6} \end{cases}$$

$$(2.8)\quad \begin{cases} y_{i+1} = y_i + h_i y_i' + \frac{h_i^2}{2!} a_{i,2} + \frac{h_i^3}{3!} a_{i,3} + \frac{h_i^4}{4!} a_{i,4} + \frac{h_i^5}{5!} y_i^{(5)} \\ y_{i+1}' = y_i' + h_i a_{i,2} + \frac{h_i^2}{2!} a_{i,3} + \frac{h_i^3}{3!} a_{i,4} + \frac{h_i^4}{4!} y_i^{(5)} \\ y_{i+1}^{(5)} = y_i^{(5)} \end{cases}$$

$$(2.9)\quad \begin{cases} y_n = y_{n-1} + h_{n-1} y_{n-1}' + \frac{h_{n-1}^2}{2!} a_{n-1,2} + \frac{h_{n-1}^3}{3!} a_{n-1,3} + \frac{h_{n-1}^4}{4!} a_{n-1,4} \\ \quad + \frac{h_{n-1}^5}{5!} y_{n-1}^{(5)} + \frac{h_{n-1}^6}{6!} a_{n-1,6} \\ y_n' = y_{n-1}' + \frac{h_{n-1}}{1!} a_{n-1,2} + \frac{h_{n-1}^2}{2!} a_{n-1,3} + \frac{h_{n-1}^3}{3!} a_{n-1,4} + \frac{h_{n-1}^4}{4!} y_{n-1}^{(5)} \\ \quad + \frac{h_{n-1}^5}{5!} a_{n-1,6} \\ y_n^{(5)} = y_{n-1}^{(5)} + h_{n-1,6} \end{cases}$$

From these equations, we have

$$(2.10)\begin{cases} a_{0,3} = 6h_0^{-3}\left(4y_1 - 4y_0 - h_0 y_1' - 3h_0' y_0' - h_0^2 y_0''\right) \\ \quad + \frac{1}{60} h_0^2 \left(2y_0^{(5)} + y_1^{(5)}\right) \\ a_{0,4} = 24h_0^{-3}(3h_0^{-1} - 3h_0^{-1} y_1 + y_1' + 2y_0') \\ \quad + 12h_0^{-2} y_0'' - \frac{1}{10} h_0 (6y_0^{(5)} - y_{1,}^{(5)}) \\ a_{0,6} = h_0^{-1}\left(h_1^{(5)} - y_0^{(5)}\right) \\ \quad . \\ \quad . \end{cases}$$

$$(2.11)\quad \begin{cases} a_{i,2} = h_i^{-2}\left(2y_i - h_i y_{i+1}' - 16 h_i y_i' - 4h_i^2 y_i''\right) \\ \quad + \frac{7}{180} h_i^3 y_i^{(5)} - \frac{1}{72} h_i^3 y_{i+1}^{(5)}, \\ a_{i,3} = 6h_i^{-3}\left(4y_{i+1} - h_i y_{i+1}' - 4y_i + 5h_i y_i' - h_i^2 y_i''\right) \\ \quad + \frac{1}{20} h_i^2 \left(3y_i^{(5)} + y_{i+1}^{(5)}\right), \\ a_{i,4} = 24h_i^{-3}\left(-3h_i^{-1} y_{i+1} + \frac{3}{2} h_i^{-1} y_{i+1}' + 3h_i^{-1} y_i + 2h_i y_i' + h_i y_i''\right) \\ \quad - \frac{h_i}{30}\left(11y_i^{(5)} - y_{i+1}^{(5)}\right), \end{cases}$$

and

$$(2.12)\begin{cases} a_{n-1,\ 2} = 3y_{n-1}'' + \frac{h_{n-1}^3}{360}\left(2083 y_{n-1}^{(5)} - 649 y_n^{(5)}\right), \\ a_{n-1,3} = 6h_{n-1}^{-3}\left(4y_n - h_{n-1} y_n' - 4y_{n-1} - 3h_{n-1} y_{n-1}' - h_{n-1}^2 y_{n-1}''\right) \\ \quad + \frac{1}{60} h_{n-1}^2 \left(11 y_{n-1}^{(5)} + y_n^{(5)}\right). \\ a_{n-1,4} = 24h_{n-1}^{-4}(-3y_n + h_{n-1} y_n' + 3y_{n-1} + 2h_{n-1} y_{n-1}' + \frac{1}{2} h_{n-1}^2 y_{n-1}'' \\ \quad - \frac{1}{30} h_{n-1}\left(5y_{n-1}^{(5)} - y_n^{(5)}\right) \\ a_{n-1,6} = h_{n-1}^{-1}\left(y_n^{(5)} - y_{n-1}^{(5)}\right) \end{cases}$$

The unique existence of the coefficients above show the unique existence of the spline function $s_\Delta(x)$ of theorem 1.This is the prove of the theorem.

3. Theorem of Convergence

Let $f(x) \in C^5(1)$, then the uniqne spline function $s_\Delta(x)$ mentioned in theorem 1, with $y_{i,}$ etc being associated with the function $f(x), ie\ y_i = f(x_{i,}), y_i = f'(x_{i,})$ etc., we have for $x \in [x_{i,}x_{i+1,}], i = 0(1)n-1,$

(3.1) $|s_\Delta^{(q)}(x) - f^{(q)}(x)| \le kq\, h^{5-q} w_5(h), q = 0(1)5$

Here we take h_i=h for all i =0(1)n-1, and denote the modulus of continuity of $f(x) \in C^5(I)$ by w_5(h).

Proof:

Let first be $x_i \le x \le x_{i+1}, i = 0(1)n-2$, from (2.4) and writing finite Taylor's sums for f(x) and is derivatives, we have

$$S_i^{(q)}(x) - f^{(q)}(x) = \frac{(x-x_i)^{5-q}}{(5-q)!}\left(a_{i,5} - f^{(5)}(x_i)\right) + \frac{(x-x_i)^{6-q}}{(6-q)!}(a_{i,6} - f^{(6)}(\varepsilon_{i,2})$$

When q = 4, 5; where $x_i < \xi_{i,q} < x_{i+1}$.

Let $x \in [x_0, x_1], then\ q = 5, i = 0(1)n$

$$s_0^{(5)}(x) - f^{(5)}(x) = (x - x_0)(a_{i,6} - f^{(6)}(x)\ )$$

$$= (x - x_0)[h_0^{-1}(y_1^{(5)} - y_0^{(5)} - f^{(6)}(x)]$$

Therefore,

$|s_0^{(5)}(x) - f^{(5)}(x)| \le w_5(h)$

Using the interpolatory conditions, we have

$|s_0^{(4)}(x) - f^{(4)}(x)| = |\int_{x_0}^{x_1}.[s_0^{(5)}(x) - f^{(5)}(x)]dx|$

$$\le |x_1 - x_0|.|s_0^{(5)}(x) - f^{(5)}(x)|$$

$$\le h\, w_5(h)$$

Again using Taylor's theorem, we have

$$|s_0''(x) - f''(x)| \le h^2 w_5(h)$$

Further,

$|s_0''(x) - f''(x)| \le |\int_{x_0}^{x_1}.\ \{s_0''(x) - f''(x)\}\mathrm{d}x|$

$$\le |x_1 - x_0||s_0''(x) - f''(x)|$$

$$\le h^3 w_5(h)$$

Similar

$$|s_0'(x) - f'(x)| \le h^4 w_5(h)$$

And $|s_0(x) - f(x)| \le h^5 w_5(h).$

This is the prove of theorem 2 for $x \in [x_{0,}x_1]$.

For $x \in [x_{i,}x_{i+1}], \quad i = (1)n-2$, we have (2.5) using Taylor's theorem

$$|s_i^{(5)}(x) - f^{(5)}(x)| = |f_{i,5}^{(5)}(x) - f^{(5)}(x)|$$

$$= |f_{i,5}^{(5)}(n) - f_{i,5}^{(5)}(x) + f_{i,5}^{(5)} - f^{(5)}(x)|$$

$$\le w_5(h)(where\ y_{i+1}^{(5)} = y_i^{(5)}.)$$

This gives the results for $x \in [x_{i,} \ x_{i+1}]$, $i = (1)n - 2$. Proof for $x \in [x_{n-1,} \ x_n]$ can be carried out on similar lines, so we omit the details.

4. Conclusion

We have taken here a (0, 1, 5) lacunary interpolation problem for which we found a quantic spline function $s(x) \in S_{n,5}^{(2)}(x)$ which interpolates the given data. Also it is shown that this spline function converges uniquely by finding error bounds. Such types of spline function can be used for solving differential equations and obtaining quatrature formula.

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