Journal of Applied Science and Education, 2024,
Vol. 04, Iss. 02, S. No. 064, pp. 1-9
ISSN (Online): 2583-1372

Function of Fractional Calculus and Relationship Between Distribution Theory and Partial Differential Equations

Chinta Mani Tiwari[1], Shilpa Pal[2]

[1,2]Department of Mathematics, Maharishi University of Information Technology, Lucknow, India
[1]cmtiwari.12@gmail.com, [2]pinki.january1998@gmail.com

How to cite this paper: C. M. Tiwari, S. Pal, "Function of Fractional Calculus and Relationship Between Distribution Theory and Partial Differential Equations," *Journal of Applied Science and Education (JASE)*, Vol. 04, Iss. 02, S. No. 064, pp 1-9, 2024.

https://doi.org/10.54060/a2zjournals.jase.64

Received: 06/05/2023
Accepted: 16/06/2024
Online First: 21/06/2024
Published: 25/07/2024

Abstract

Partial Differential Equations are fundamental mathematical tools used to model physical phenomena across various disciplines such as physics, engineering, and economics. Solving PDEs often involves the concept of distributions, which extends the classical notion of functions to more generalized objects. This paper provides an introduction to PDEs, outlines the basic concepts of distribution theory, and discusses their interplay in solving PDEs. and distributional derivatives its convergence with distributional solution with space $\mathcal{D}'(\Omega)$ *its extended form with weak derivatives.*

Keywords

Partial Differential Equations (PDEs), Distribution theory, Burger's equation and Holmgren's theorem, Distributional Derivatives.

1. Introduction

Calculus has long been the foundation of mathematical analysis, offering strong tools for modelling and understanding a wide range of phenomena in scientific fields [1-3]. But when it comes to handling complicated systems that display non-local or memory-dependent behaviours, classical calculus, which is based on integer-order differentiation and integration, frequently fails. The idea of differentiation and integration is extended to non-integer orders in fractional calculus, a field of mathematical analysis, as a result of this limitation. Numerous disciplines, including physics, engineering, biology, economics, and finance, have used fractional calculus. Among other things, its capacity to explain fractal geometry, viscoelastic materials, and anomalous diffusion has attracted a lot of interest from scientists looking for more precise models of processes that occur in the actual world [4-7]. Of particular note, fractional differential equations have become essential instruments for the analysis of complicated dynamics that defy standard models.

Furthermore, our knowledge of how to precisely define solutions for a variety of differential equations has advanced significantly thanks to the connection between distribution theory and partial differential equations (PDEs) [7-14]. Laurent Schwartz developed distribution theory in the middle of the 20th century, and it offers a framework for expanding the concept of functions to include generalized functions or distributions in addition to basic smooth functions. With this addition, singularities and discontinuities that frequently occur in the setting of PDEs can be treated, allowing for a more comprehensive formulation and analysis of solutions.

In this work, we investigate the role of fractional calculus and its importance when dealing with complex systems that exhibit non-local behaviors or memory effects. In addition, we explore the complex relationship that exists between distribution theory and partial differential equations, looking at how the basic ideas of distribution theory improve our comprehension and analysis of PDEs and ultimately open the door to more thorough mathematical modelling techniques.

1.1. Introduction to Partial Differential Equations (PDEs)

Partial Differential Equations (PDEs) are equations involving functions of multiple variables and their partial derivatives. They are used to describe various physical phenomena where the rates of change of quantities depend on multiple independent variables. Examples of systems that can be described by PDEs include heat conduction, fluid dynamics, quantum mechanics, and electromagnetism [15-22].

A general form of a first-order linear PDE is:

$$a_1(x,y)\frac{\partial u}{\partial x}+a_2(x,y)\frac{\partial u}{\partial y}=f(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}) \tag{1.1}$$

Similarly, a second-order linear PDE can be represented as:

$$a(x,y)\frac{\partial^2 u}{\partial x^2}+2b(x,y)\frac{\partial^2 u}{\partial x\,\partial y}+c(x,y)\frac{\partial^2 u}{\partial y^2}=f(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial x\,\partial y},\frac{\partial^2 u}{\partial y^2}) \tag{1.2}$$

1.2. Introduction to Distribution Theory

Distribution theory, also known as generalized function theory, extends the concept of functions to a larger class of objects called distributions. These objects allow for the rigorous treatment of functions that are not necessarily smooth or integrable. A distribution is defined as a linear functional on a space of test functions, which are typically smooth and have compact support [23].

The space of test functions is often denoted by $D(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$. A distribution T acts on a test function $\varphi \in D(\Omega)$ by the integral:

$$< T,\varphi >= \int_\Omega T(x)\varphi(x)dx \tag{1.3}$$

Distributions are used to generalize the notion of derivatives. For example, the Dirac delta function $\delta(x)$ is a distribution that represents a point mass at the origin. It is defined by its action on a test function:

$$< \delta, \varphi >= \varphi(0) \tag{1.4}$$

1.3. Solving PDEs using Distribution Theory

Distribution theory provides a powerful framework for solving PDEs, especially those with generalized solutions. The notion of weak solutions, which are solutions in the distributional sense, allows for the treatment of PDEs with discontinuous coefficients or singular sources.

Consider the Poisson equation in two dimensions:

$$\nabla^2 u = f(x, y) \tag{1.5}$$

In distributional form, the Poisson equation becomes:

$$\int_\Omega \nabla^2 u\varphi \, dx = \int_\Omega f\varphi \, dx \tag{1.6}$$

Using integration by parts and the definition of distributions, we can rewrite this equation as:

$$-\int_\Omega \nabla u \cdot \nabla\varphi \, dx = \int_\Omega f\varphi \, dx \tag{1.7}$$

This equation holds for all test functions $\varphi \in D(\Omega)$, which leads to the weak form of the Poisson equation. By considering appropriate test functions, one can obtain solutions to PDEs in the distributional sense.

2. Methodology

When we talked about Burger's equation and Holmgren's theorem, we remarked on the topic of distributional solutions. The finding that for a linear partial P differential operator was crucial in demonstrating the latter, $P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ and a classical solution of $Pu = w$ in Ω (open subset of $\mathbb{R}^n$) with $D^\beta u = 0$ on $\partial\Omega$ for $|\beta| \leq m - 1$, we have the integration by parts formula

$$\int_\Omega w \cdot v dx = \int_\Omega \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha(x) v) \cdot u dx \tag{2.1}$$

valid for all $v \in C_0^\infty(\Omega)$. The formula (1) makes sense for u merely continuous or even $u \in L^1_{loc}(\Omega)$ and it is natural to declare $u \in L^1_{loc}(\Omega)$ a "weak" or a "distributional" solution of $Pu = w$ if it satisfies (1) for all $v \in C_0^\infty(\Omega)$.

Generally, with any (real-or) complex valued function $f: \Omega \to \mathbb{C}$, continuous on $\Omega \subset \mathbb{R}^n$ open (or even $u \in L^1_{\mathrm{loc}}(\Omega)$), we can associate its integrals against test functions by defining

$$f[\phi] := \int_\Omega f(x)\phi(x) dx \tag{2.2}$$

for $\phi \in \mathcal{D} := C_0^\infty(\Omega)$. Note that

- $f[\phi]$ is a linear functional on the space of test functions $\mathcal{D}$
- $f[\phi]$ is well defined for $f \in L^1_{loc}(\Omega)$
- the definition supports the idea of "smeared averages" from physics: If f is an observable like a velocity or a temperature, you will never be able to determine its value at a point but only averaged over a small interval (finite detector size).
- If f is continuous, then $f[\phi]$ determines f uniquely (so in some sense we don't lose anything by considering $f[\phi]$ instead of f)

- One can differentiate f in the sense of distributions by defining

$D_k f[\phi] := -f[D_k \phi]$ (2.3)

which agrees with the usual derivative if $f \in C^1$ by the standard integration by parts formula. Therefore, linear partial differential operators act naturally on the functional $f[\phi]$.

3. The space D' (Ω)

We broaden our view further and consider general linear functional on the space $\mathcal{D}$ of test-functions (of which those arising by integration against an L^1 function, the $f[\phi]$ above, are a particular example). We shall introduce s notion of continuity of such functionals below (which is desirable if we would like to keep the interpretation as physical observables). This notion of continuity its most easily formulated via sequential continuity.

Definition. We say that $\phi_n \in \mathcal{D}$ converges to ϕ in $\mathcal{D}$ if

- there is a compact set K such that all ϕ_{n} vanish outside K
- there is a $\phi \in \mathcal{D}$ such that for all $\alpha \in \mathbb{N}^d$ we have $\theta^\alpha \phi_n \to \theta^\alpha \phi$ uriformnly in x.

Definition. A distribution is a linear functional $\ell: \mathcal{D}(\Omega) \to \mathbb{C}$, which is continuous in the sense that if ϕ_n converges to ϕ in $\mathcal{D}$, the $\ell(\phi_n) \to \ell(\phi)$. The vectorspace of distributions in denoted $\mathcal{D}'(\Omega)$.

Example. Each continuous (or L^1_{loc}) function generates a distribution via (2). Such distributions are called regular distributions. Not every distribution is regular, as the next example shows.

Example. The distribution $\delta_\ell[\phi] = \phi(\xi)$ is not regular. Indeed, the formula $\int dx g(x)\phi(x) = \phi(\xi)$ would imply that $g \in L^1_{\mathrm{loc}}$ vanishes everywhere (modulo a set of measure 0). This example also makes it intuitive to talk about the support of a distribution: If $f[\phi] = g[\phi]$ for all ϕ with support in $\omega \subset \Omega$, we Il say that the two distributions agree in ω.

The above notion of continuity may be cumbersome to check in practical applications. However, we have the following

3.1. Proposition: The function $\ell: \mathcal{D}(\Omega) \to \mathbb{C}$ belong to $\mathcal{D}'(\Omega)$ if and only if for every compact subset $K \subset \Omega$ there is an integer $n(K, \ell)$ and a $c \in \mathbb{R}$ such that for all $\phi \in \mathcal{D}(\Omega)$ with support in K we have

$|\ell[\phi]| \leq c \parallel \phi \parallel_{C^n}$ with $\parallel \phi \parallel c^n = \sum_{|\alpha| \leq n} \max_x |\theta^\alpha \phi|$ (3.1)

Proof. The "if" follows immediately from the estimate (3.1). For "only if" suppose that (3.1) was violated for some compact set K. Then we can find for this K a sequence ϕ_n with $\|\phi_n\| C^n = 1$ and $|\ell[\phi_n]| \geq n$ (otherwise the estimate (3.1) would hold with $= N$). But then $\psi_n = n^{-1/2} \phi_n$ is a sequence converging to wero in $\mathcal{D}$, while $|\ell[\psi_n]| \geq \pi^{1/2}$ does not go to zero. Contradiction.

If there is a c such that (3.1) holds, ℓ is said to be of order n on K. IF ℓ is of order n on every compact subset $K \subset \Omega$, the ℓ is of order n on Ω.

Example. Any regular distribution is of order 0. The Dirac delta of Example is also under zero.

Example. The principal value distribution

$$\ell[\phi] := \lim_{\kappa \to 0} \int_{|z| > c} \frac{\phi(x)}{x} = P.V. \int \frac{\phi(x)}{x} \quad (3.2)$$

is a distribution of under 1 (near 0 at least; away from zero it is order 0). The proof is an exercise. Hint: Taylor-expand ϕ near 0 and use the symmetry of the integral.

4. Distributional Derivatives

We can define the distributional derivative $D_k f$ as the distribution

$D_k f[\phi] = -f[D_k\phi]$ or more generally $D^\alpha f[\phi] = (-1)^{|\alpha|} f[D^\alpha \phi$ |. (4.1)

You should check that this indeed defines a distribution.
Compute $D_k \delta_\epsilon |\phi|$.
Therefore, we can apply a linear operator P of order m to a distribution $u[\phi]$ vir
$Pu[\phi] = u[P^t \phi]$. (4.2)

Below we will be particularly interested in distributional solutions of

$$Pu = \delta_f - u[P^t \phi].$$

A distribution u satisfying the equation is called a fundamental solution with pole ξ for the operator P.

5. Relation with weak derivatives

If f is a regular distribution, i.e.

$$f[\phi] = \int_\Omega \phi(x) f(x) dx \tag{5.1}$$

for some $f \in L^1_{\text{loc}}(\Omega)$ then it may be that the distributional derivative is again a regular distribution. In other words, there could be a $g \in L^1_{\text{loc}}$ such that

$$D^k f[\phi] := -\int D^k\phi(x) f(x) dx = \int \phi(x) g(x) dx \tag{5.2}$$

holds for any ϕ in $\mathcal{D}$. In this case, we say that f has $g = D^k f$ as its weak derivative. Using an argument similar to one already used above, it is easy to show that the weak derivative, if it exists, is unique.
To see that not every function ($\equiv$ regular distribution) has a weak derivative consider the example of the step function $H: \mathbb{R} \to \mathbb{R}$ defined as

$$H(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

This is clearly in L^1_{loc} but the distributional derivative is easily seen to be the delta distribution in view of the following computation:

$$D_x H[\phi] = -\int_{-\infty}^{\infty} D_z\phi(x) H(x) dx = \int_0^\infty - D_z\phi(x) H(x) = \phi(0). \tag{5.3}$$

Using the notion of a weak derivatives one can define various notions of "weak solutions" to a PDE, which will typically require some number of weak derivatives to exist.

Some questions:

a. Show that the distributional derivative of $\log|x|$ is $P \cdot V \cdot \frac{1}{x}$.

b. (Fritz John, 3.6 (3)) Show that the function

$$u(x_1, x_2) = \begin{cases} 1 & \text{for } x_1 > \xi_1, x_2 > \xi_2 \\ 0 & \text{for all other } x_1, x_2 \end{cases}$$

defines a fundamental solution with pole (ξ_1, ξ_2) of the operator $L = \frac{\theta^2}{\partial z_1 \delta x_2}$ in the $x_1 x_2$-plane.

6. More on Distributions (non-examinable)

6.1. Convergence of Distributions

Definition. A sequence of distributions $\ell_n \in \mathcal{D}'(\Omega)$ converges to $\ell \in \mathcal{D}'(\Omega)$ if and only if for every teat function $\phi \in \mathcal{D}(\Omega)$ we have

$$\ell_n[\phi] \to \ell[\phi] \tag{6.1}$$

with the usual notion of convergence in C. We will write $\ell_n \sim \ell$ to denote this convergence and say that ℓ_n convergences "weakly" or "in the sense of distributions" to ℓ.

Exercise: Show that the sequence of (regular) distributions $n^2 e^{inz}$ converges weakly to zero as $\pi \to \infty$.

Exercise: Let $j \in \mathcal{D}(\mathbb{R}^{\mathrm{d}})$ with $\int_{\mathbb{R}^d} j(x) d^d x = 1$. Define $j_k(x) = \epsilon^{-d} j\left(\frac{z}{c}\right)$. Show that $j_c \to \delta_{\mathrm{p}}$.

The previous example is remarkable as it shows that the non-regular delta distribution can be approximated by function in $\mathcal{D}$. In fact, any element in $\mathcal{D}'$ can be approximated in this way: The space $\mathcal{D}$ is dense in $\mathcal{D}'$. This can be used to extend (uniquely) the usual operations of calculus (differentiation, translation, convolution) to $\mathcal{D}'$, which is very useful for PDE.

6.2. Extending Calculus from D (Ω) to D' (Ω)

We will run in a rather informal way through the main ideas of extending various operations of calculus from test functions to distributions. The key is Proposition 2 in the Appendix of Rauch's book, which we repeat below. Remember we already have a notion of convergence in $\mathcal{D}$, Definition.

Proposition: Suppose $L: \mathcal{D}(\Omega_1) \to \mathcal{D}(\Omega_2)$ is a linear, sequentially continuous map. Suppose in addition that there is a linear, sequentially continuous map $L^{\mathrm{t}}: \mathcal{D}(\Omega_2) \to \mathcal{D}(\Omega_1)$ which is the transpose of L in the sense that

$$\int_{\Omega_2} L\phi \cdot \psi = \int_{\Omega_1} \phi \cdot L^{\mathrm{t}}\psi \text{ holds for all } \phi \in \mathcal{D}(\Omega_1), \psi \in \mathcal{D}(\Omega_2). \tag{6.2}$$

Then the operator L extends to a sequentially continuous map of $\mathcal{D}'(\Omega_1) \to \mathcal{D}'(\Omega_2)$ given by

$$L(\ell)[\psi] = \ell[L^t\psi] \text{ for all } \ell \in \mathcal{D}'(\Omega_1), \psi \in \mathcal{D}(\Omega_2). \tag{6.3}$$

You can find the (very easy) proof in Rauch's book or do it yourself. This simple proposition allows us to define the following operations on distributions

- multiplication of a distribution with a C^∞ function $f \in C^\infty(\Omega)$. Since on test functions the transpose is itself (multiplication by f), we have
$$(f - \ell)[\psi] = \ell[f \cdot \psi]$$
- translation of a distribution (say $\Omega = \mathbb{R}^d$ so that we don't have to keep track of domains). Since $(\tau_y f)(x) = f(x - y)$ on test functions has transpose τ_{-y} we have $\tau_{\mathrm{y}}(\ell)[\psi] = \ell[\tau_{-\mathrm{g}}\psi]$ on distributions.
- reflection of a distribution: $\Re(\ell)|\psi\rangle = \ell[\Re\psi]$, ss the transpose of $(\Re f)(x) = f(-x)$ on test functions is itself.
- derivative of a distribution (we already did that!)
- convolution of a distribution with a C^∞ function (see below)

The remarkable point of convolving a distribution with a smooth function is that the result is actually a smooth function. You will prove this below. This is very useful and can be used to show that we can approximate any element in $\mathcal{D}'(\mathbb{R}^d)$ by elements in $\mathcal{D}(\mathbb{R}^d)$.

Let $\Omega = \mathbb{R}^d$ and $f \in \mathcal{D}(\mathbb{R}^d)$. For $g \in \mathcal{D}(\mathbb{R}^n)$, the convolution of g with f is defined ss

$$(f * g)(x) := \int_{\mathbb{R}^4} f(x - y) g(y) dy = (g * f)(x). \tag{6.4}$$

Exercise: Show that (on test functions) the transpose of convolution with f is convolution with $\Re f$

Therefore we define

$$(f * \ell)[\psi] = \ell[\Re f * \psi] \tag{6.5}$$

7. Applications of Fractional Calculus

Fractional calculus involves derivatives and integrals of non-integer (fractional) orders. Its applications span numerous areas:

a. Physics

Anomalous Diffusion: Fractional calculus models sub-diffusion and super-diffusion processes, which cannot be described by classical diffusion equations.

Viscoelastic Materials: Fractional derivatives describe the complex behavior of materials that exhibit both viscous and elastic characteristics.

b. Engineering

Control Theory: Fractional order controllers (like PI^λD^μ controllers) offer more flexible and robust control strategies.

Signal Processing: Fractional Fourier transform provides a tool for time-frequency analysis, extending the classical Fourier transform.

c. Biology and Medicine

Modeling of Biological Systems: Fractional models describe the memory and hereditary properties of various biological tissues.

Tumor Growth: Fractional differential equations can model the growth dynamics of tumors more accurately than integer-order models.

d. Finance

Option Pricing Models: Fractional calculus helps model markets with memory effects, improving the accuracy of financial models

7.1. Relationship between Distribution Theory and Partial Differential Equations

Distribution theory, or the theory of generalized functions, provides a framework for extending the concept of functions and derivatives, enabling the treatment of functions that are not necessarily smooth. This theory is crucial for dealing with PDEs, particularly when solutions involve singularities or are not well-defined in the classical sense.

a. Fundamental Solutions

The Dirac delta function, a key object in distribution theory, is often used as a fundamental solution to linear PDEs, serving as a Green's function to express solutions in terms of source terms.

b. Weak Solutions

In many PDEs, classical solutions may not exist. Distribution theory allows for the formulation of weak solutions, where equations are satisfied in an integral sense. This approach is vital in fields such as fluid dynamics and quantum mechanics.

c. Regularization Techniques

Distributions help in regularizing PDEs with singular coefficients or initial/boundary data, making it possible to analyze and solve otherwise intractable problems.

d. Fourier Transform and PDEs

Distribution theory extends the Fourier transform to a broader class of functions, facilitating the solution of PDEs in the frequency domain.

7.2. Combining Fractional Calculus and Distribution Theory in PDEs

Fractional calculus and distribution theory can be combined to tackle complex PDEs, offering a robust mathematical framework:

a. Fractional Differential Equations (FDEs)

Distribution theory aids in defining and solving FDEs, especially when solutions are not smooth. The generalized functions can represent solutions with discontinuities or singularities.

b. Green's Functions for FDEs

Extending the concept of Green's functions to fractional PDEs often involves distributions, providing integral representations of solutions.

c. Stochastic Processes

Fractional and distributional methods are used in modeling stochastic processes governed by fractional PDEs, applicable in fields such as finance and physics.

Fractional calculus and distribution theory enrich the study of PDEs by providing tools to model and analyze phenomena with memory effects, singularities, and non-local interactions. These advanced mathematical concepts extend the boundaries of traditional calculus and enable the solution of more complex and realistic problems in various scientific and engineering disciplines.

8. Conclusion

Partial Differential Equations (PDEs) play a crucial role in modelling various physical phenomena. Distribution theory provides a powerful framework for solving PDEs, especially those with generalized solutions. By extending the notion of functions to distributions, one can tackle PDEs with discontinuous coefficients or singular sources. Understanding the interplay between PDEs and distribution theory is essential for advancing in the fields of mathematics, physics, and engineering.

References

[1]. K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. John Wiley & Sons, 1993.

[2]. I. Podlubny and L. Schwartz, “Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution, and Some of Their Applications,” *Academic Press*, 1998.

[3]. L. Hörmander, *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*. Springer-Verlag, 1990.

[4]. F. J. Almgren, *Plateau’s Problem: An Invitation to Varifold Geometry*. New York: Addison-Wesley, 1966.

[5]. F. J.Almgren, and J. E. Taylor, “The geometry of soap films and soap bubbles”, *ScientificAmerican*, 235, July 1976.

[6]. H. Anton, *Calculus with Analytic Geometry, JohnWiley and Sons*. New York, 1980.

[7]. D. F. Bailey, W. E. Boyce, R. C. DiPrima, and M. Braun, “Elementary differential equations and boundary value problems,” *Am. Math. Mon.*, vol. 84, no. 8, p. 664, 1977.

[8]. P. R. Garabedian, “Partial Differential Equations”, *second edition, Chelsea Publishing,New York*, 1986.

[9]. P. Gillett, *Calculus and Analytic Geometry, D.C. Heath and Co*. Lexington, Massachusetts, 1981.

[10]. K. E. Gustafson, “Introduction to Partial Differential Equations and Hilbert Space Methods*”, New York: JohnWiley*, 1987.

[11]. K. Karamcheti, Principles of Ideal-Fluid Aerodynamics*, Robert E. Robert E; Malabar, Florida: Krieger Publishing Company*, 1980..

[12]. J. D. Logan, “Applied Mathematics: A Contemporary Approach”, *New York: John Wiley*, 1987.

[13]. P. Moon and D. E. Spencer, “Field Theory Handbook”, *New York: Springer-Verlag*, 1961.

[14]. R. Osserman, “A Survey of Minimal Surfaces”, *Dover*, New York, 1986.

[15]. S. D. Peckham, “Self-similarity in the three-dimensional geometry and dynamics of large river basins”, *PhD thesis, University of Colorado, Boulder*, 1995.

[16]. J. W. W. and M. R. Spiegel, “Mathematical handbook of formulas and tables,” *Math. Comput.*, vol. 23, no. 108, p. 886, 1969.

[17]. D. Zwillinger, “Handbook of Differential Equations”, *Academic Press, New York*, 1989.

[18]. C. M. Tiwari, “A note on Dirac delta function,” *The Aligarah bulletin of Mathematics*, vol. 25, pp. 11–15, 2006.

[19]. C. M. Tiwari, “Neutrix product of three distributions,” *The Aligarah bulletin of Mathematics*, vol. 25, no. 1, pp. 33–38, 2007.

[20]. C. M. Tiwari, “A commutative group of generalized function”,” *Journal of Indian Academy of Mathematics*, vol. 29, no. 1, pp. 71–78, 2007.

[21]. C. M. Tiwari, “Neutrix product of two distribution using….”. *Journal of Indian Academy of Mathematics*.ISSN no. 0970-5120 vol.30, no.1, pp 1-5, 2008.

[22]. C. M. Tiwari, “The Neutrix product of the distribution x……”. International journal of scientific reseaerch Engineering and Management (IJSREM),” vol. 1, pp. 6693–6695, 2023./doi.org/10.22214/ijraset.2023.53222

[23]. C. M. Tiwari, “Generalized function and distribution…”,” *International Journal for Scientific Research Innovations*, vol. 1, pp. 1–6, 2023.