Journal of Applied Science and Education, 2024,
Vol. 04, Iss. 02, S. No. 066, pp. 1-10
ISSN (Online): 2583-1372

Neutrix Convolution of Ultra-Distributions Product and Distributions on C^{∞}- Manifolds

Chinta Mani Tiwari[1], Vaibhav Joshi[2]

[1,2]Department of Mathematics, Maharishi University of Information Technology, Lucknow, India
[1]cmtiwari.12@gmail.com, [2]vaibhavjoshi189@gmail.com

How to cite this paper: C. M. Tiwari, V. Joshi, "Neutrix Convolution of Ultra-Distributions Product and Distributions on C∞-Manifolds," *Journal of Applied Science and Education (JASE)*, Vol. 04, Iss. 02, S. No. 066, pp 1-10, 2024.

https://doi.org/10.54060/a2zjournals.jmss.66

Received: 04/05/2024
Accepted: 20/06/2024
Online First: 25/07/2024
Published: 25/07/2024

Abstract

The absence of a general definition for convolutions and products of distribution is one of the issues with distribution theory. It is discovered in quantum theory and physics that certain convolutions and products such as $\frac{1}{x}+\delta$ are in usea description of the term "product of distributions" and a list of sample product results using a particular delta sequence $\delta_n(x) = C_m n^m \rho(n^2 r^2)$ in an m-dimensional space. The Fourier transform is applied to $D'(m)$ and the exchange formula for defining ultradistribution convolutions in $Z'(m)$ in terms of products of distributions in $D'(m)$. We are going to demonstrate a theorem that says that for any items $\tilde{f}$ and $\tilde{g}$ in $Z'(m)$, the neutrix convolution $\tilde{f} \otimes \tilde{g}$ exists in $Z'(m)$ if and only if the product $f \circ g$ exists in $D'(m)$. Some convolutional findings are derived using van der Corput's neutrix calculus. Let V'(M) be a smooth m-manifold M's space of distributions, each specified by an assemblage of 'compatible' ordinary distributions (components) displayed on the charts of some C^{∞} on M. Drawing on van der Corput's concept of neutrix limitations, we expand the definition of the neutrix distribution product in this context. onto the space V'(M). We establish the existence of certain theorems regarding the neutrix distribution product in the space V'(M) under various assumptions on the neutrix product of the constituents.

Keywords

Distribution theory, m-dimensional space, ultradistribution convolutions, neutrix calculus, delta sequence.

1. Introduction

First, review the definition of generalized functions (distributions) that we accept for any arbitrary smooth m-dimensional real manifold. From now on, we shall refer to this manifold as a "manifold." For every manifold M, where some C^{∞}. $\{\kappa_i, M_i\}_{i\in I}$ on it, we shall use the notation: $\widetilde{M}_i = \kappa_i(M_i) \subseteq \mathbf{R}^m$, $M_{ij} = M_i \cap M_j$ and $\kappa_{ij} := \kappa_i(\kappa_j^{-1}) : \kappa_j(M_{ij}) \to \kappa_i(M_{ij})$ for the (coordinate) diffeomorphic maps of class C^∞ of open sets in $\mathbf{R}^m(i,j\in I)$. Further, for arbitrary open subset U of M we shall denote: $U_i = U\cap M_i, U_{ij} = U_i\cap U_j$ and $\widetilde{U}_i = \kappa_i(U_i)\subseteq \mathbf{R}^m(i,j\in I)$. Later on we shall often need the following.

Theorem 1. Let $\kappa\colon U_1\to U_2$ be a C^∞-diffeomorphic map of open sets in $\mathbf{R}^m$. Then there is a unique continuous linear map of the distribution spaces $\kappa^*\colon \mathcal{D}'(U_2)\to\mathcal{D}'(U_1)\colon F\mapsto \kappa^*F$ (pull-back of F by) coinciding with the composition of functions $F(\kappa(x))$ whenever F is in $C^0(U_2)$ and it holds for any test-function ϕ in $\mathcal{D}(U_1)$

$$\langle \kappa^*F,\phi\rangle = \langle F,\psi\rangle,\ \text{where } \psi=\phi(\kappa^{-1})|\det\ D\kappa^{-1}|\in\mathcal{D}(U_2). \tag{1.1}$$

Further, for each function f in $C^\infty(U_1)$

$$\kappa^*(F\cdot f) = \kappa^*F\cdot f(\kappa)$$

and for any open subset V_2 of U_2 and $V_1=\kappa^{-1}(V_2)$ open in U_1

$$\kappa^*(F\mid v_2) = (\kappa^*F)\mid v_1,$$

Where the restriction $F|_v$ is defined by $\langle F\mid v,\psi\rangle = \langle F,\bar{\psi}\rangle$ for each ψ in $\mathcal{D}(U)$ and $\bar{\psi}=\{\psi$ on $V, 0$ on $U\setminus V\}$.

Proof. For an arbitrary distribution F in $D'(U_2)$, let $\{F_n(x)\}$ be a sequence of infinitely differentiable functions in U_2 converging weakly to F, as $n\to\infty$. Then, on making the substitution $t=\kappa(x)$, we have for any ϕ in $\mathcal{D}(V_1)$:

$$\int_{U_1} F_n(\kappa(x))\phi(x)dx = \int_{U_2} F_n(t)\phi(\kappa^{-1}(t))|\det\ D\kappa^{-1}|dt = \int_{U_2} F_n(t)\phi(t)dt \tag{1.2}$$

With a test-function ψ in $\mathcal{D}'(U_2)$ defined as in. Now taking the weak limits as $n\to\infty$, we see that the sequence $\{F_n(\kappa(x))\}$ converges to the unique distribution κ^*F in $\mathcal{D}'(U_1)$ given by above equation. Moreover, the map κ^* : $\mathcal{D}'(U_2)\to\mathcal{D}'(U_1)\colon F\mapsto\kappa^*F$ is linear, continuous and coincides with the ordinary composition of functions in $C^0(U_2)$, by its construction. Further, equations readily follow on noting that:

$$\int_{U_1} (F_n\cdot f)(\kappa(z))\phi(x)dx = \int_{U_1} F_n(\kappa(z))\cdot f(\kappa(z))\phi(z)dx \tag{1.3}$$

Both expressions in this equation clearly lead to the same distribution in $\mathcal{D}'(U_1)$, when we make the substitution $t=\kappa(z)$ and pass to the weak limits, as $n\to\infty$.

In order to prove, we get the following chain of equations for arbitrary sequence $\{F_n(x)\}$ weakly converging to F, on making the due substitutions

$$\begin{aligned}
\int_{V_1} (F_{\mathrm{n}}\mid V_2)(\kappa)\phi(x)dx &= \int_{V_2} (F_{\mathrm{N}}\mid V_2)(t)\psi(t)dt\ \ [\text{with } \psi \text{ defined as in } (1)\big]\\
\int_{V_1} (F_{\mathrm{n}}\mid V_2)(\kappa)\phi(x)dx &= \int_{U_2} F_n(t)\bar{\psi}(t)dt\ [\bar{\psi}=\psi \text{ on } V_2, 0 \text{ on } U_2\setminus V_2]\\
\int_{V_1} (F_{\mathrm{n}}\mid V_2)(\kappa)\phi(x)dx &= \int_{U_1} (F_{\mathrm{N}}(\kappa(x)))\bar{\phi}(x)dx\ [\bar{\phi}=\phi \text{ on } U_1, 0 \text{ on } U_1\setminus V_1]\\
\int_{V_1} (F_{\mathrm{n}}\mid V_2)(\kappa)\phi(x)dx &= \int_{V_1} (F_n(\kappa(x)))\Big|_{V_1}\phi(x)dx.
\end{aligned} \tag{1.4}$$

Since the restriction map $R_V\colon F\mapsto F|_V$ is linear and continuous, we obtain on passing to the weak limits as $n\to\infty$, that $\langle\kappa^*(F\mid v_2),\phi\rangle = \langle(\kappa^*F)\mid v_1,\phi\rangle$ for any ϕ in $\mathcal{D}(V_1)$. This completes the proof of the theorem.

In the following, let $\rho(x)$ be a fixed infinitely differentiable function with the properties

(i) $\rho(x)=0,\ |x|\geq 1,$

(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)dx = 1$.

We define the function $\delta_n(x)$ by $\delta_n(x) = n\rho(nx)$ for $n = 1,2,\cdots$. It is clear that $\{\delta_n\}$ is a sequence of infinitely differentiable functions converging to the Dirac delta-function δ.

Now let D be the space of infinitely differentiable functions with compact support. If f is an arbitrary distribution in D', we define the function f_n by $f_n = f * \delta_n$. It follows that $\{f_n\}$ is a sequence of infinitely differentiable functions converging to f.

The following definition was given by B.

Definition 1. Let f and g be distributions in D' and let $g_n = g * \delta_n$. We say that the neutrix product $f \circ g$ of f and g exists and equals h if

$$N - \lim_{n\to\infty} (f g_n, \phi) = (h, \phi) \tag{1.5}$$

For all ϕ in D, where N is the neutrix (see van der Corput [4]) having domain $N' = \{1,2,\cdots,n,\cdots\}$ and range N'' the real numbers with negligible functions finite linear sums of the functions:

$$n^{\lambda} \ell n^{r-1} n,\ \ell n^{r} n (\lambda > 0, r = 1,2,\cdots) \tag{1.6}$$

And all functions of n which converge to zero as n tends to infinity.

Let $D'(m)$ be the space of distributions defined on the space $D(m)$ of infinitely differentiable functions of the variable $x = (x_1, x_2, \cdots, x_m)$ with compact support.

In order to give a definition for the neutrix product $f \circ g$ of two distributions f and g in $D'(m)$, we attempt to define a δ-sequence in $D(m)$ by putting:

$$\delta_n(x_1, x_2, \cdots, x_m) = \delta_n(x_1) \cdots \delta_n(x_m), \tag{1.7}$$

where δ_n is defined as above. However, this definition is very difficult to use for distributions in $D'(m)$ which are functions of r, where $r = (x_1^2 + \cdots + x_m^2)^{1/2}$. Therefore, an alternative definition was introduced.

From now on we let $\rho(s)$ be a fixed infinitely differentiable function defined on $R^+ = [0,\infty)$ having the properties
(i) $\rho(s) = 0, s \geq 1$,
(ii) $\rho(s) \geq 0$.
Define the function $\delta_n(x)$, with $x \in R^m$, by
$\delta_n(x) = C_m n^m \rho(n^2 r^2)$
for $n = 1,2,\cdots$, where C_m is a constant such that
$\int_{R^m} \delta_n(x)dx = 1$.

Definition 2. Let f and g be distributions in $D'(m)$ and let

$$g_n(x) = (g * \delta_n)(x) = (g(x-t), \delta_n(t)) \tag{1.8}$$

where $t = (t_1, t_2, \cdots, t_m)$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to h on the open interval (a, b), where $a = (a_1, \cdots, a_m)$ and $b = (b_1, \cdots, b_m)$, if

$$N - \lim_{n\to\infty} (f g_n, \phi) = (h, \phi) \tag{1.9}$$

for all test functions ϕ is $D(m)$ with support contained in the interval (a,b).

2. Fourier Transform on $\mathrm{D}'(\mathrm{m})$

As in Gel'fand and Shilov, we define the Fourier transform of a function ϕ in $D(m)$ by

$$F(\phi)(\sigma)=\psi(\sigma)=\int_{R^m} \phi(x)e^{i(x,\sigma)}dx, \tag{2.1}$$

where (x,σ) denotes $x_1\sigma_1+\cdots+x_m\sigma_m$.

The bounded support of $\phi(x)$ makes it possible for ψ to be continued to complex values of its argument $=(s_1,\cdots,s_m)=(\sigma_1+i\tau_1,\cdots,\sigma_n+i\tau_m)$:

$$\psi(s)=\int_{R^m} \phi(x)e^{i(x,s)}dx. \tag{2.2}$$

Our new function $\psi(s)$, defined on C^m, in the space of functions of m complex variables, is continuous and analytic in each of its variable s_k. If $\phi(x)$ vanishes for $|x_k|>a_k, k=1,\cdots,m$, then $\psi(s)$ satisfies the inequality

$$\left|s_1^{\sigma_1}\cdots s_m^{q_m}\psi(\sigma_1+i\tau_1,\cdots,\sigma_m+i\tau_m)\right|\leq C_q\exp\,(a_1|\tau_1|+\cdots+a_m|\tau_m|). \tag{2.3}$$

Conversely, every entire function $\psi(s_1,\cdots,s_m)$ satisfying the above inequality is the Fourier transform of some $\phi(x_1,\cdots,x_m)$ in $D(m)$ which vanishes for $|x_k|>a_k, k=1,2,\cdots,m$.

The set of all entire analytic functions $Z(m)$ with the property (1) is in fact the space

$$F(D(m))=\{\psi:\exists\phi\in D(m) \text{ such that } F(\phi)=\psi\}. \tag{2.4}$$

Convergence in $Z(m)$ is defined vis corrvergense in $D(\mathrm{m})$: a sequence $\{\psi_{\mathrm{n}})$ tends to zero in $Z(\mathrm{m})$ If the sequerce $[\phi_2\}$ tends to $zera$ in $D(m)$, wtere $F(\phi_2)=\psi_{\mathrm{p}}$. The Fourier transform $\}$ of a discrbution in $D'(\mathrm{m})$ is an diradistribution in $Z'(\mathrm{m})$, i.t. a costimuous linear fincticeal $\alpha a=(\mathrm{m})$. In is delined by Purveral's equatisn

$$(\mathrm{b},\phi)=2\pi(f,\phi),\ \phi\in D(\mathrm{m}). \tag{2.5}$$

3. Convolution in $Z'(\mathrm{m})$

We present the Fourier transform to define a convolution product in Z’(m)

$f(\delta_n) and\ \delta_n$.

(where $G_n(x)=C_nn^n\rho(v^2r^2)$) and write

$$r_0(a)=F(\varepsilon_s)(a)$$

which is a function in $Z(m)$ for $n=1,2,\cdots$.

From Parsevals equation:

$$\begin{aligned}(\tau_*,\psi)=2\pi(\delta_n,\phi)\Rightarrow 2\pi(\delta,\phi)=2\pi\phi(0)\quad &=2\pi\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(\rho)d\sigma\\ &=(1,\psi)\end{aligned} \tag{3.1}$$

where $\phi=\phi$.

Therefore (r_{n}) is a sequence in $Z(\mathrm{m})\subset Z'(\mathrm{m})$ converging to 1 in $Z'(\mathrm{m})$.

Now $\tilde{f}$ be an arbitrary ultra-distribution in $Z'(\mathrm{m})$. Then there exists a distribution f in $D(m)$ such that $\hat{\gamma}=F(f)$. Setting $\tilde{f}_n=F(f*G_{\mathrm{n}})=F(f_{\mathrm{n}})$, we have

$$(f,v)=2\pi(f,\phi)\rightarrow 2\pi(f,\phi)=(f,\psi)\ n\rightarrow\infty \tag{3.2}$$

Lemma 1. Let g be an arbitrary ultradistribution in $Z'(m)$ and let $g_n=F(g*\xi_n)$ Then the function

$$\theta_n(v)=\{G_3(\rho),\psi[\sigma+v)\} \tag{3.3}$$

is in $Z(m)$ for all ψ in $Z(m)$.
Indeed.

$$\begin{aligned} \theta_n(\nu) &= (F(g_n), F(e^{in}\phi(z))(\sigma)) \\ \theta_n(\nu) &= 2\pi(g_n, e^{nn}\phi(z)) = 2\pi F(g_n\phi)(\nu). \end{aligned} \quad (3.4)$$

We now modify the definition for the convolution product of two distributions in $D'(\mathrm{m})$ given by Gelfind end Stilov:

Definition 3. Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $Z'(m)$ such that the function $(\tilde{g}(\sigma), \psi(\sigma+\nu))$ is in $Z(m)$ for all ψ in $Z(m)$. Then the convolution product $\tilde{f} * \tilde{g}$ is defined by

$$((\tilde{f} * \tilde{g})(\sigma), \psi(\sigma)) = (\tilde{f}(\nu), (\tilde{g}(\sigma), \psi(\sigma+\nu))) \quad (3.5)$$

for all ψ in $Z(m)$.

It follows that $\tilde{f} * \tilde{g}$ exists if $g\phi$ is in $D(m)$. (This condition is not always true for all $g \in D'(m)$. If $\tilde{g} \in Z(m)$, then $g\phi \in D(m)$.) Indeed $(\tilde{g}(\sigma), \psi(\sigma+\nu)) = 2\pi(g, e^{iz\nu}\phi(x)) = 2\pi F(g\phi)(\nu)$, (3.6)
where $\tilde{g} = F(g)$ and $\psi = F(\phi)$.

The following theorem then holds:
Theorem 1. Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $Z'(m)$ and suppose that the convolution product $\tilde{f} * \tilde{g}$ exists. Then
Proof. If $F(\phi) = \psi$, we have

$$\psi'(\sigma) = F(ix\phi(x))(\sigma).$$

Hence $Z'(m)$ is closed under differentiation.
Certainly

$$\begin{aligned} ((\tilde{f} * \tilde{g})', \psi) &= -(\tilde{f} * \tilde{g}, \psi') = -\left(\tilde{f}(\nu), (\tilde{g}(\sigma), \psi'(\sigma+\nu))\right) \\ ((\tilde{f} * \tilde{g})', \psi) &= \left(\tilde{f}(\nu), (\tilde{g}'(\sigma), \psi(\sigma+\nu))\right) = (\tilde{f} * \tilde{g}', \psi) \end{aligned} \quad (3.7)$$

for all ψ in $Z(m)$. Equation follows. From the fact that if $F(\phi)$, we get

$$\psi'(\sigma+\nu) = F(ix\phi(x)e^{ix\nu})(\sigma).$$

It follows that:

$$\begin{aligned} (\tilde{g}(\sigma), \psi'(\sigma+\nu)) &= 2\pi(g(x), ix\phi(x)e^{1x\nu}) \\ (\tilde{g}(\sigma), \psi'(\sigma+\nu)) &= 2\pi\frac{d}{d\nu}(g(x), \phi(x)e^{1x\nu}) \\ (\tilde{g}(\sigma), \psi'(\sigma+\nu)) &= \frac{d}{d\nu}(\tilde{g}(\sigma), \psi(\sigma+\nu)). \end{aligned} \quad (3.8)$$

Hence,

$$((\tilde{f} * \tilde{g})', \psi) = (\tilde{f}'(\nu), (\tilde{g}(\sigma), \psi(\sigma+\nu))) = (\tilde{f}' * \tilde{g}, \psi) \quad (3.9)$$

for all ψ in $Z(m)$ and Equation follows.
Note that $\tilde{f}' \neq F(f')$ is general.

We now note that if $\tilde{f}$ and $\tilde{g}$ are arbitrary ultradistributions in $Z'(m)$, then the convolution product $\tilde{f} * \tilde{g}_n$ always exists by the above definition since by Lemma 1, $(\tilde{g}_n(\sigma), \psi(\sigma+\nu))$ is in $Z(m)$ for all ψ in $Z(m)$. This leads us to the following definition.

Definition 4. Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $Z'(m)$ and let $\tilde{g}_n = \tilde{g}\tau_n$. Then the neutrix convolution product $\tilde{f} \otimes \tilde{g}$ is defined to be the neutrix limit of the sequence $\{\tilde{f} * \tilde{g}_n\}$, provided the neutrix limit $\tilde{h}$ exists in the sense that

$$N - \lim_{n\to\infty} \left(\tilde{f} * \tilde{g}_n, \psi\right) = (\tilde{h}, \psi) \text{ for all } \psi \text{ in } Z(m); \tag{3.10}$$

Definition 4 is indeed a generalization of Definition 3, since if the convolution product $\tilde{f} * \tilde{g}$ exists by Definition 3, then $(\tilde{g}(\sigma), \psi(\sigma+\nu)) \in Z(m)$, i.e., $g\phi \in D(m)$ for all $\phi \in D(m)$. This implies $g \in C^\infty(m)$.
Therefore $(\tilde{g}_n(\sigma), \psi(\sigma+\nu)) = 2\pi F(g_n\phi)(\nu)$ converges to $(\tilde{g}(\sigma), \psi(\sigma+\nu))$ in $Z(m)$. This is because $g_n\phi \to \phi$ (if $f \in C^\infty$, then $f_n\phi$ (where $f_n = f * \delta_n$) converges to f_ϕ uniformly on the support of ϕ) in $D(m)$, and $N - \lim_{n\to\infty}\left(\tilde{f} * \tilde{g}_n, \psi\right) = \left(\tilde{f} * \tilde{g}, \psi\right)$ for all ψ in $Z(m)$.
The following theorem holds for the neutrix convolution product.

Theorem 2. Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $Z'(m)$ and suppose that their neutrix convolution product exists. Then the neutrix convolution product $\tilde{f} \otimes \tilde{g}$ exists and

$$(\tilde{f} \otimes \tilde{g})' = \tilde{f}' \otimes \tilde{g}.$$

Proof. We have

$$\left((\tilde{f} * \tilde{g}_n)', \psi\right) = \left(\tilde{f}' * \tilde{g}_n, \psi\right) = -\left(\tilde{f} * \tilde{g}_n, \psi'\right) \tag{3.11}$$

and it follows that

$$N - \lim_{n\to\infty} \left(\tilde{f}' * \tilde{g}_n, \psi\right) = -N - \lim_{n\to\infty} \left(\tilde{f} * \tilde{g}_n, \psi\right) = -\left(\tilde{f} \otimes \tilde{g}, \psi'\right) \tag{3.12}$$

for arbitrary ψ in $Z(m)$. The result of the theorem follows.
Note that $(\tilde{f} \otimes \tilde{g})' = \tilde{f} \otimes \tilde{g}'$ iff $N - \lim_{n\to\infty}\left(\tilde{f} * (\tilde{g}\tau_n), \psi\right) = 0$ for all ψ in $Z(m)$. (3.13)
We now prove our main result, the exchange formula.

Theorem 3. Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $Z'(m)$. Then the neutrix convolution product $\tilde{f} \otimes \tilde{g}$ exists in $Z'(m)$ iff the neutrix product $f \circ g$ exists in $D'(m)$ and the exchange formula

$$\tilde{f} \otimes \tilde{g} = 2\pi F(f \circ g)$$

is then satisfied.
Proof. Let $\psi = F(\phi)$ be an arbitrary function in $Z(m)$ and let

$$\Theta_n(\nu) = (\tilde{g}_n(\sigma), \psi(\sigma+\nu)) = 2\pi F(g_n\phi)(\nu). \tag{3.14}$$

Then on using Parseval's equation we have

$$\left(\tilde{f}(\nu), \Theta_n(\nu)\right) = 2\pi\left(\tilde{f}(\nu), F(g_n\phi)(\nu)\right) = (2\pi)^2(f g_n, \phi). \tag{3.15}$$

If the neutrix convolution product $\tilde{f} \otimes \tilde{g}$ exists then

$$\begin{aligned}(\tilde{f} \otimes \tilde{g}, \phi) &= N - \lim_{n\to\infty}\left(\tilde{f}(\nu), \Theta_n(\nu)\right) = (2\pi)^2 N - \lim_{n\to\infty}(f g_n, \phi)\\ (\tilde{f} \otimes \tilde{g}, \phi) &= (2\pi)^2(f \circ g, \phi) = 2\pi(F(f \circ g), F(\phi)).\end{aligned} \tag{3.16}$$

The neutrix product $f \circ g$ therefore exists and the exchange formula is satisfied. Conversely, the existence of the neutrix product $f \circ g$ implies the existence of the neutrix convolution product and the exchange formula.

4. Some Results

The following Fourier transforms of the functions r^{λ} and $\Delta^k \delta(x)$ were given

$$F\left(r^{\lambda}\right) = 2^{\lambda+m}\pi^{m/2}\frac{\Gamma\left(\frac{\lambda+m}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}\rho^{-\lambda-m} \quad (4.1)$$

where $\lambda \neq -m, -m-2, \cdots$ and $\rho = \sqrt{\sum_{i=1}^{m}\sigma_i^2}$, and

$$F\left[P\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\right)f(x)\right] = P(-is_1, \cdots, -is_m)F(f).$$

Hence it follows that

$$F(\Delta^k \delta(x)) = \rho^{2k}F(\delta) = \rho^{2k}, \quad (4.2)$$

where Δ denotes the Laplace operator.

Theorem 4. The neutrix convolution products $\rho^{2k-m} \otimes 1$ and $\rho^{2k-1-m} \otimes 1$ exist and

$$\rho^{2k-m} \otimes 1 = \frac{\Gamma(k)2^{k-m+1}\rho^{2k}}{\Gamma\left(\frac{m-2k}{2}\right)\pi^{m/2-1}k!m(m+2)\cdots(m+2k-2)} \quad (4.3)$$

for $k = 1,2, \cdots, \left[\frac{(m-1)}{2}\right]$ and

$$\rho^{2k-1-m} \otimes 1 = 0$$

for $k = 1,2, \cdots, \left[\frac{m}{2}\right]$.

The next theorem now gives a natural sufficient condition for the existence of the neutrix distribution product in the space $\mathcal{D}'(M)$.

Theorem 5. Given the distributions $F = \{F_i\}_{i\in I}$ and $G = \{G_i\}_{i\in I}$ on a manifold M with an atlas $\{\kappa_i, M_i\}_{i\in I}$ on it, suppose the neutrix product $F_i \circ G_i$ exist (in $\mathcal{D}'_m$) and is equal to H_i on the whole domain $\widetilde{M}_i$ for all i in I. Then the neutrix product $F \circ G$ exist in $\mathcal{D}'(M)$ and is equal to $H = \{H_i\}_{i\in I}$ on the whole manifold M.

Proof. Consider the distribution H on M defined by the collection $\{H_i = F_i \circ G_i\}_{i\in I}$ of distributions in $\mathcal{D}'\left(\widetilde{M_\iota}\right)$. Then for each i in I equations holds. Taking the pull-back map of the component H_i by κ_{ij}, we get for any i in I

$$\kappa_{ij}^* H_i = \kappa_{ij}^*(F_i \circ G_i) = \kappa_{ij}^* F_i \circ \kappa_{ij}^* G_i = F_j \circ G_j = H_j$$

Each equation here holds on the whole domain $\widetilde{M}_i \subseteq \mathbf{R}^m$, except the third one that holds on $\kappa_j\left(M_{ij}\right) \subseteq \widetilde{M}_i$. Thus, we get exactly the consistency between the components H_i and H_j for arbitrary i and j in the index set I. According to the Lemma, we have thus defined a unique distribution H in $\mathcal{D}'(M)$. Clearly, it satisfies Definition 5 with an open set U coinciding with M (and all $\widetilde{U}_i = \widetilde{M}_i$). The proof of the theorem is complete.

We note that the sufficient condition set up by this theorem would apply to a variety of particular neutrix products in $\mathcal{D}'(M)$ since most of the neutrix distribution products proved so far to exist are each equal to some distribution on the whole space.

A further refinement of this existence theorem is given below. We first introduce the following notation. Any open set U in given manifold M and atlas $\{\kappa_i\}_{i\in I}$ on it can be viewed as submanifold of M with an inclusion map id $\mathrm{id}_U: U \to M: x \mapsto x$ and atlas $\{U_i, \kappa_i^U = \kappa_i|_{U_i}\}_{i\in I}$. Thus, applying Definition 1, we can define the space of distributions on U, which we shall denote by $\mathcal{D}'_M(U)$ (with an index ' M ' indicating the parent manifold).

Theorem 6. Given the distributions $F = \{F_i\}_{i\in I}$ and $G = \{G_i\}_{i\in I}$ on a manifold M with atlas $\{\kappa_i\}_{i\in I}$ and an arbitrary open set U in M, suppose the neutrix product $F_i \circ G_i$ exists (in $\mathcal{D}'_m$) and is equal to H_i on the open set $\vec{U}_i$ for any i in I. Then there is a unique distribution $K = \{K_i\}_{i\in I}$ on the submanifold U of M, such that $K_i = H_i|_{\tilde{U}_i}$ for all i in I.

Proof. Consider the distribution K on the submanifold U of M defined by the collection $\{K_i = H_i|_{\tilde{U}_i}\}_{i\in I}$ of distributions in $\mathcal{D}'(\tilde{U}_i)$. We have

$F_i \circ G_i = K_i$ on $\tilde{U}_i$ for each i in I, and therefore equation holds. Now we show that $\{K_i\}_{i\in I}$ is 'well-defined' distribution on U. Indeed, for any i and j in I, the following chain of equations for the pull-back map by κ_{ij}^U can be obtained:

$$\begin{aligned}
(\kappa_{ij}^U)^* K_i = (\kappa_{ij}^U)^*(H_i|_{\tilde{U}_i}) \quad &= (\kappa_{ij}^U)^*((F_i \circ G_i)|_{\tilde{U}_i}) \\
&= (\kappa_{ij}^U)^*\left((F_i|_{\tilde{U}_i}) \circ (G_i|_{\tilde{U}_i})\right) \\
&= \left((\kappa_{ij}^U)^*(F_i|_{\tilde{U}_i})\right) \circ \left((\kappa_{ij}^U)^*(G_i|_{\tilde{U}_i})\right) \\
&= \left(((\kappa_{ij}^U)^* F_i)\big|_{\tilde{U}_i}\right) \circ \left(((\kappa_{ij}^U)^* G_i)\big|_{\tilde{U}_i}\right) \\
&= \left(F_j\big|_{\tilde{U}_j}\right) \circ \left(G_j\big|_{\tilde{U}_j}\right) \text{ [by (4)]} \\
&= H_j\big|_{\tilde{U}_j} = K_j.
\end{aligned} \tag{4.4}$$

Each equation here holds on the whole $\tilde{U}_j$, except for that obtained by holding on $\kappa_j(M_{ij}) \cap \tilde{U}_j = \kappa_j(U_{ij})$. We therefore have: $(\kappa_{ij}^U)^* K_i = K_j$ on the set $\kappa_j(U_{ij})$ for all i and j in I, and it follows from the Lemma that the collection $\{K_i\}_{i\in I}$ defines a unique distribution K in the space $\mathcal{D}'_M(U)$. This completes the proof of the theorem.

Finally, we shall employ the following canonical definitions. For a given distribution $F = \{F_i\}_{i\in I}$ in $\mathcal{D}'(M)$ and an open set U in M consider the collection $\{G_i = F_i|_{\tilde{U}_i}\}_{i\in I}$ (we can put $G_i = 0$ if $U \cap M_i$ is empty). In the above results, their elements satisfy the consistency condition and thus they define a unique distribution in $\mathcal{D}'_M(U)$, that can equally be denoted by $F|_U$. Further, we have this definition for the equality of distributions on M.

Now F = G on an open set *U* if *F \u= G \u* in $\mathcal{D}'_M(U)$. (4.5)

5. Application and Mathematical Use of Neutrix Convolution on C^∞-Manifolds

The concept of neutrix convolution extends the classical convolution operation to broader classes of functions and distributions, such as ultra-distributions. When dealing with C^∞-manifolds, the convolution operation must be defined in a way that respects the manifold's smooth structure. This allows for advanced applications in mathematical physics, partial differential equations, and signal processing on manifolds.

The neutrix convolution of two distributions f and g, denoted by $(f * g)$, is defined as a limit of regularized convolutions. For ultra-distributions, this involves extending the notion of convolution beyond classical distributions, allowing for more singular objects. Mathematically, if f and g are ultra-distributions, their neutrix convolution is defined as:

$$(f * g)(x) = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f(t) g(x - t) \eta_\epsilon(t) dt \quad (4.6)$$

where η_ϵ is a sequence of regularizing functions (often chosen as a delta-sequence).

Distributions on C^∞-Manifolds

A C^∞-manifold M is a topological space that locally resembles $\mathbb{R}^n$ and has a globally defined smooth structure. Distributions on M are continuous linear functionals on the space of smooth test functions $C_c^\infty(M)$.

Application: Signal Processing on C^∞-Manifolds, in signal processing, convolution is a fundamental operation for filtering and analyzing signals. When signals are defined on a C^∞-manifold, the neutrix convolution can be used to handle singularities and extend classical methods to more complex geometries.

Example Application: Consider a C^∞-manifold M representing the surface of a curved sensor array. Let f be a signal recorded by the sensors and g a filter function (both modeled as distributions on M). To apply the neutrix convolution on M :

1. Local Representation: Cover M with coordinate charts $(U_\alpha, \varphi_\alpha)$, where U_α are open sets in M and $\varphi_\alpha: U_\alpha \to \mathbb{R}^n$ are diffeomorphisms.

2. Local Convolution: In each chart, represent f and g locally as distributions on $\mathbb{R}^n$. Compute the local neutrix convolution $(f_\alpha * g_\alpha)(x)$ using the definition for ultra-distributions.

3. Global Assembly: Use partition of unity $\{\psi_\alpha\}$ subordinate to the cover $\{U_\alpha\}$ to piece together the local convolutions into a global distribution on M:

$$(f * g)(x) = \sum_\alpha \psi_\alpha(x) (f_\alpha * g_\alpha)(\varphi_\alpha(x)) \quad (4.7)$$

This process ensures the convolution respects the manifold's smooth structure.

Mathematical use, consider a specific C^∞-manifold M, such as the 2-sphere S^2. Let f and g be ultra-distributions on S^2 representing some physical quantities (e.g., temperature distribution and heat kernel).

1. Choose Regularizing Sequence: Select η_ϵ as a sequence of mollifiers on S^2, adapted to the manifold's geometry.

2. Compute Convolution: For each ϵ, compute the regularized convolution:

$$(f *_\in g)(x) = \int_{S^2} f(y) g(exp_x^{-1}(y)) \eta_\epsilon(d(x, y)) dV(y)$$

where exp_x^{-1} is the inverse exponential map and d(x,y) is the geodesic distance.

3. Take Neutrix Limit: Define the neutrix convolution as the limit:

$$(f * g)(x) = \log_{\in \to \infty} (f *_\in g)(x)$$

This application allows for advanced analysis of signals on curved surfaces, with potential applications in geophysics, medical imaging, and other fields requiring signal processing on non-Euclidean domains.

6. Conclusion

The Neutrix convolution of distributions on C∞-manifolds and ultra-distributions offers an effective framework for comprehending and working with singularities in a variety of mathematical applications. We are able to have a better understanding of the interaction between smooth and unique structures by expanding the concept of convolution to spaces of ultra-distributions. This framework provides an abundant environment to investigate distribution behavior under diverse transformations and operations, opening new avenues for research and applications in fields including quantum field theory, harmonic analysis, and partial differential equations. Research on Neutrix convolution is still ongoing and productive, with great potential for expanding our knowledge of the complex interrelationships between smoothness and singularity in mathematical analysis.

References

[1]. J. G. van der Corput, "Introduction to the neutrix calculus," Journal d'Analyse Mathématique, vol. 7, pp. 291-398, 1959.

[2]. B. Damyanov, "On the sheaf of generalized functions over a C^{∞}manifold," in Proceedings, 1993, pp. 83-88.

[3]. B. Fisher, "The non-commutative neutrix product of distributions," Mathematische Nachrichten, vol. 108, pp. 117-127, 1982.

[4]. B. Fisher, "The non-commutative neutrix product of the distributions and $\delta^{(p)}(x)$" Indian Journal of Pure and Applied Mathematics, vol. 14, pp. 1439-1449, 1983.

[5]. B. Fisher, "On defining a non-commutative product of distributions in R variables," Journal of Natural Science and Mathematics, vol. 31, pp. 95-102, 1991.

[6]. W. Güttinger, "Products of improper operators and the renormalization problem of quantum field theory," Progress of Theoretical Physics, vol. 13, no. 6, pp. 612-626, 1955.

[7]. L. Hörmander, The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis, Berlin: Springer, 1983.

[8]. J. H. Bremermann, Distributions, Complex Variables, and Fourier Transforms, Reading, MA: Addison-Wesley, 1965.

[9]. Q. Y. Chen, "A survey of the budding nonlinear theory of distributions," Journal of Mathematics (in Chinese), vol. 1, pp. 127-133, 1981.

[10]. C. K. L. Lee and B. Fisher, "Examples of the neutrix product of distributions on R," Radovi Matematički, vol. 6, no. 1, pp. 129-137, 1990.

[11]. J. G. van der Corput, "Introduction to the neutrix calculus," Journal d'Analyse Mathématique, vol. 7, pp. 291-398, 1959.

[12]. B. Fisher, "On defining the convolution of distributions," Mathematische Nachrichten, vol. 106, pp. 261-269, 1982.

[13]. I. M. Gelfand and G. E. Shilov, Generalized Functions I, New York: Academic Press, 1964.

[14]. C. M. Tiwari, "A note on Dirac delta function," The Aligarh Bulletin of Mathematics, vol. 25, no. 1, pp. 11-15, 2006.

[15]. C. M. Tiwari, "Neutrix product of three distributions," The Aligarh Bulletin of Mathematics, vol. 25, pp. 33-38, 2007.

[16]. C. M. Tiwari, "A commutative group of generalized functions," Journal of the Indian Academy of Mathematics, vol. 29, no. 1, pp. 71-78, 2007.

[17]. C. M. Tiwari, "Neutrix product of two distributions using tensor product," Journal of the Indian Academy of Mathematics, vol. 30, no. 1, pp. 1-5, 2008.

[18]. C. M. Tiwari, "The Neutrix product of the distribution x^n," International Journal of Scientific Research in Engineering and Management, vol. 1, no. 1, pp. 6693-6695, 2023, doi: 10.22214/ijraset.2023.53222.

[19]. C. M. Tiwari, "Generalized function and distribution theory," International Journal of Scientific Research and Innovation, vol. 1, pp. 1-6, 2023.