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Study of Some Properties on Almost Para Contact Metric Manifolds with Certain Connection

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Abstract

Many differential geometers studied different types of manifolds with a semi-symmetric metric connection. In the present paper, we study semi-symmetric nonmetric connections on an almost para-contact manifold in relation to semi-symmetric nonmetric connections. In another section of the work, we have studied the curvature tensor and Nijenhuis tensor.

Keywords

Semi-symmetric nonmetric connection, almost para contact metric manifold, Nijenhuis tensor.

Subject classification

53C05, 53D15, 53B15, 53C25

1. Introduction

Semi-symmetric metric connection has been studied by various mathematicians including Ram Nivas [7], Srivastava [5], Imai, and S.I. Hussain [4]. I. Sato defined and studied para contact manifolds. K.D. Singh and Rakeshwar Singh have studied semi-symmetric metric connection on an almost para-contact manifold [9]. Recently Nirmala S. Agashe and others have defined the motion of semi symmetric nonmetric connection in a Riemannian manifold [1].

2. Preliminaries

Let M^n be an n-dimensional real differentiable manifold equipped with a C^∞ (1,1) tensor field f, a C^∞ vector field T and a C^∞ 1-form A satisfying

$$` (a) \overline{\overline{X}} = X - A(X)T \qquad where \overline{X} = f(X) \;\; (b) A(T) = 1 \tag{1.1}$$

Then the structures (f, T, A) on M^n is said to be an almost para contact structure manifold. It can be verified that on M^n the following holds.

$$(a) \overline{T} = 0 \qquad (b) A(\overline{X}) = 0 \qquad (c) rank(f) = n - 1 \tag{1.2}$$

An almost para contact manifold M^n with structure (f, T, A) always admits a positive definite Riemannian metric g which satisfies

$$(a) g(\overline{X}, \overline{Y}) = g(X, y) - A(X)A(Y) \;\; (b) g(X, T) = A(X) \tag{1.3}$$

M^n endowed with sucha metric g is called almost para contact metric manifold with structure (f, T, A) from (1.3) it follows that

$$g(\overline{\overline{X}}, \overline{\overline{Y}}) = g(\overline{X}, \overline{Y}) \tag{1.4}$$

If we put

$$F(X, Y) = g(\overline{X}, Y) \tag{1.5}$$

then we get

$$(a) F(X, Y) - F(X, Y) = 0 \;\; (b) F(X, \overline{Y}) - F(\overline{X}, Y) = 0 \;\; (c) F(T, Y) = 0$$
(1.6)

A linear connection ∇ is said to be semi-symmetric connection on the almost para contact manifold M^n if its torsion tensor

$$S(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

satisfies the formula

$$S(X, Y) = A(Y)X - A(X)Y \tag{1.7}$$

∇ is said to be semi-symmetric non meyric with respect to the associated Riemannian metric g if

$$(\nabla_X g) = -A(Y)g(X, Z) - A(Z)g(X, Y) \tag{1.8}$$

We define ∇ to be semi-symmetric nonmetric f-connection if in addition to (1.7) and (1.8) ∇ satisfies

$$(\nabla_X f) = 0 \tag{1.9}$$

Suppose ∇ is a Riemannian connection on M^n, then we can always put

$$\nabla_X Y = D_X Y + u(X, Y) \tag{1.10}$$

u being tensor of (1,2) type satisfying

$$g(u(X, Y), Z) + g(u(X, Z), Y) = A(Y)g(X, Z) + A(Y)g(X, Z) \tag{1.11}$$

Obviously, we have

$$S(X,Y) = u(X,Y) - u(Y,X) \tag{1.12}$$

Nirmala S. Agashe and others has expressed the value of u(x, y) in terms of S and S', both being tensors of type (1, 2) as follows [1]

$$u(X,Y) = \frac{1}{2}\big(S(X,Y) + S'(X,Y) + S'(Y,X)\big) + g(X,Y)T \tag{1.13}$$

where

$$g(S(Z,X),Y) = g(S'(X,Y),Z) \tag{1.14}$$

It can be verified that

$$S'(X,Y) = A(X)Y - g(X,Y)T \tag{1.15}$$

and

$$u(X,Y) = A(Y)X$$

Thus, we get

$$\nabla_X Y = D_X Y + A(Y)X \tag{1.16}$$

It is easy to verify that

$$(a) S'(Y,X) = u(X,Y) + g(X,Y)T \;\; (b) g\big(S(X,Y,T)\big) = 0 \;\; (c) S(X,Y) = X^{-} \;\; (d) S'(Y,X) = u(X,T) + A(X)T \;\; (e) S'(X,Y) - S'(Y,X) = S(X,Y) \tag{1.17}$$

Theorem 1. *In an almost para contact manifold M^n, the torsion tensor of the semi-symmetric non-metric connection satisfies the following identities* $(a) S(X,T) = X^{-}$ $(b) S(X^{-},Y) = A(Y)X - A(X)A(Y)T$ $(c) S(X^{-},Y) + S(X,Y^{-}) = S(X,Y)$ $(d) S(X^{-},Y) - S(X^{-},T) = 0$ $(e) A\big(S(X,Y)\big) = 0$ $(f) S(X,Y) = S(X,Y)^{-}$ (1.18)

Now we will establish certain identities among the (0,3) type tensors defined by [5]

$$S'(X,Y,Z) = g\big(S(X,Y),Z\big) \;\; and \;\; u'(X,Y,Z) = g\big(u(X,Y),Z\big) \tag{1.19}$$

or equivalently

$$S'(X,Y,Z) = (g(Y,T)\; g(X,T)\; g(Y,Z)\; g(X,Z)\;)$$

and

$$u'(X,Y,Z) = (g(Y,T)\; g(Z,T)\; g(X,Y)\; g(X,Z)\;)$$

Theorem 2. *The following relations hold in an almost metric manifolds* $(a) u'(X,Y^{-},Z^{-}) = S'(X^{-},Y^{-},Z) = 0$ $(b) u'(X,Y,Z) = S'(Z,Y,X)$ $(c) u'(X,Y,Z) = -u(X,Z,Y)$ $(d) S'(X,Y,Z) = -S'(Y,X,Z)$ $(e) S'(X,Y,Z) - S'(X,Z,Y) = u'(X,Y,Z)$ $(f) u'(X^{-},Y,Z) - u'(X,Y^{-},Z) - u'(X,Y,Z^{-}) = 0$ $(g) u'(X^{-},Y,Z) - u'(X,Z,Y) = 0$ (1.20)

Theorem 3. *The connection ∇ , D and the (0,3) type tensor u' of the almost para contact metric manifold (F,T,A,g) are related by the following*

$$(a) (\nabla_X F)(Y,Z) = (D_X F)(Y,Z) = (D_X F)(Y,Z) + u'(X,Y^{-},Z) - u'(X,Y,Z^{-}) \tag{1.21}$$

$$(b) (\nabla_X F)(Y,Z) = (D_X F)(Y^{-},Z^{-}) \tag{1.22}$$

This proof is easy consequence of (1.5) and (1.6) (a).

Corollory 1. *It follows from 2 that*
$(\nabla_X F)(Y,Z) = (D_X F)(Y,Z)$ *iff* $u'(X,Y^{-},Z) = u'(X,Y,Z^{-}$

Theorem 4. $\boldsymbol{\nabla_X Y^{-} = D_X Y^{-} - F(X,Y)T}$

2. The Curvature Tensor

We denote by R and K. The curvature tensor of the semi-symmetric non-metric connection ∇ and the Riemannian connection D respectively, i.e,

$$(a) R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \ \ and \ \ (b) K(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \tag{2.1}$$

Then we state the following theorem

Theorem 5. *The two-curvature tensor are related by the following relation*

$$R(X,Y)Z = K(X,Y)Z + A(D_Y Z)X - A(D_X Z)Y - A(Z)S(X,Y) + X\big(A(Z)\big)Y - Y\big(A(Z)\big)X \tag{2.2}$$

3. The Nijenhuis Tensor

In this section we study the Nijenhuis tensor n relation to the semi-symmetric non-metric connection and establish various identities involving it. The Nijenhuis tensor is defined as

$$(a) N(X,Y) = [X^{-},Y^{-}] \ \ and \ \ (b) \ N(X,Y) = [X^{-},Y^{-}] + [X,Y]^{-} - [X,Y]^{-} - [X,Y^{-}]^{-} - A([X,Y])T \tag{3.1}$$

If we put

$$B(X,Y) = [X,Y]^{--} + [X,Y] \ \ and \ \ W(X,Y) = [X^{-},Y] + [X,Y^{-}] \tag{3.2}$$

Then (3.1) (a) reduces to

$$N(x,Y) = B(x,Y) - W(X,Y)$$

Further if we put

$$(a) B(X,Y,X) = g\big(B(X,Y),Z\big) \ \ (b) W(X,Y,X) = g\big(W(X,Y),Z\big) \ \ (c) N(X,Y,X) = g\big(N(X,Y),Z\big) \tag{3.3}$$

Then it is evident from the definitions that

$$N(X,Y,X) = B(X,Y,X) - W(X,Y,X) \tag{3.4}$$

Theorem 6. *The Nijenhuis tensor N defined on* M^n *with the Riemannian connection D satisfies the following identity*
$N(X,Y) = (D_{X^-} f)(Y) - (D_{Y^-} f)(X) - (D_X f^{-}(Y) + (D_Y f^{-}(X)$ (3.5)

Theorem 7. *B(X,Y) defined by (3.2)(a) satisfies the following equation*

$$B(X,Y) = \nabla_{X^-} Y^{-} - \nabla_{Y^-} X^{-} + \nabla_X Y - A([X,Y]) - S(X,Y) \tag{3.6}$$

Remark 1:
let ∇ be a semi-symmetric non-metric f-connection over M^n, then from (3.5) we have

$$B(X,Y) = [X,Y] + A([X,Y]) + (\nabla_{X^-} Y - \nabla_{Y^-} X)^{-} \tag{3.7}$$

Theorem 8. *An almost paracontact metric structure with semi-symmetric non-metric f-connection has vanishing Nijenhuis tensor*

Proof:
from (3.5) and (1.16) we have

$$N(X,Y) = \nabla_{X^-} fY - f\nabla_{X^-} Y - \nabla_X fY^- + f\nabla_X Y^- + \nabla_Y fX^- - f\nabla_Y X^- = \nabla_{X^-} f)(Y) - (\nabla_{X^-} f) - (\nabla_X f)(Y)^- + (\nabla_Y f)(X)^-$$

In view of ∇ being semi-symmetric non-metric f-connection the Nijenhuis vanishes.

4. Conclusion

In this paper, we have concluded that in an almost para contact metric, the connection ∇ , D and (0,3) type tensor u' are related by-

$$(a)(\nabla_X F)(Y,Z) = (D_X F)(Y,Z) = (D_X F)(Y,Z) + u'(X,Y^-,Z) - u'(X,Y,Z^-) \;\; (b)(\nabla_X F)(Y,Z) = (D_X F)(Y^-,Z^-)$$

The two-curvature tensor of semi-symmetric non-metric connection∇ and Riemannian connection D are related as $R(X,Y)Z = K(X,Y)Z + A(D_Y Z)X - A(D_X Z)Y - A(Z)S(X,Y) + X\big(A(Z)\big)Y - Y\big(A(Z)\big)X$

And, also in view of being semi-symmetric non-metric f-connection, the Nijenhuis Tensor vanishes.

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